ON A COMMON GENERALIZATION OF KOSZUL DUALITY AND TILTING EQUIVALENCE

DAG MADSEN

ABSTRACT. We propose a new definition of Koszulity for graded algebras where the degree zero part has finite global dimension, but is not necessarily semi-simple. The standard Koszul duality theorems hold in this setting. We give an application to algebras arising from multiplicity free blocks of the BGG category $\mathcal{O}$.

INTRODUCTION

Koszul algebras have traditionally been required to be graded algebras with semi-simple degree zero part, but Green, Reiten, and Solberg in [GRS] defined a notion of Koszulity for more general graded algebras where the degree zero part is allowed to be arbitrary finite dimensional. The purpose of their work was to develop a unified approach to Koszul duality and tilting equivalence. Koszulity of an algebra Λ was defined with respect to a module T, leading to the name T-*Koszul algebras*. In [Mad], we gave an account of the T-Koszul algebras of Green, Reiten, and Solberg from a derived category perspective. In this subject the right concepts seem to have been found, but in both papers mentioned the basic definitions have been a little too elaborate. We propose here a new definition of T-Koszul algebras that is more elementary and accessible.

Definition (4.1.1). Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra with $\operatorname{gldim} \Lambda_0 < \infty$, and let T be a graded Λ-module concentrated in degree zero. We say that Λ is T-*Koszul* if both of the following conditions hold.

    (i) T is a tilting Λ_0-module.
    (ii) T is graded self-orthogonal as a Λ-module.

We prove that the new definition is equivalent to the one given in [Mad], provided $\operatorname{gldim} \Lambda_0 < \infty$ (Theorem 4.1.5). As a consequence the three main theorems for Koszul algebras hold in this setting, namely, each T-Koszul algebra has a T-Koszul dual algebra which is the extension algebra of T (Theorem 4.2.1(a)), the T-Koszul dual of the T-Koszul dual algebra is isomorphic to the original algebra (Theorem 4.2.1(b)), and there is a duality between categories of T-Koszul modules over T-Koszul dual algebras (Theorem 4.3.1). In subsection 4.4 we comment on the situation when $\operatorname{gldim} \Lambda_0 = \infty$.

New results in this paper include: a proof that that ungraded and graded extension algebras of T are isomorphic even when Λ is allowed to be non-noetherian (Corollary 3.1.3), a much simplified account of T-Koszul modules (Proposition 4.1.4 and Theorem 4.3.1), a proof that under certain finiteness conditions T-Koszul dual algebras have equivalent bounded categories (Theorem 4.3.4), and a new description of so-called Δ-filtered modules over certain quasi-hereditary Koszul algebras (Corollary 5.3.3).

We present an application to the representation theory of semi-simple Lie algebras that might be of particular interest. There is a T-Koszul duality acting behind the scenes in the paper [DM], and it can be given the following surprisingly simple formulation.

Corollary (to Theorem 5.3.1). *Let A_λ be a basic algebra corresponding to a block $\mathcal{O}_\lambda$ of category $\mathcal{O}$ for a semi-simple Lie algebra. Let Δ denote the direct sum of all Verma modules in $\mathcal{O}_\lambda$. Let $E_\lambda = [\bigoplus_{i \geq 0} \mathrm{Ext}^i_{A_\lambda}(\Delta, \Delta)]^{\mathrm{op}}$. Suppose A_λ is multiplicity free. Then $A_\lambda \cong [\bigoplus_{i \geq 0} \mathrm{Ext}^i_{E_\lambda}(D\Delta, D\Delta)]^{\mathrm{op}}$ as ungraded algebras. Furthermore, if A_λ and E_λ are given the Ext-grading, then there is an equivalence of triangulated categories $\mathcal{D}^b(\mathsf{gr}\, A_\lambda) \to \mathcal{D}^b(\mathsf{gr}\, E_\lambda)$.*

In section 1 we recall some basic notions and facts about graded algebras and modules. In section 2 we discuss tilting equivalence and Koszul duality, highlighting the similarities between the two theories. In section 3 we look at graded self-orthogonal modules and the functors they define between derived categories. In section 4 we discuss the new definition of T-Koszul algebras and prove all the Koszul duality theorems. In section 5 we give an application to certain quasi-hereditary Koszul algebras, in particular multiplicity free blocks of category $\mathcal{O}$.

1. PRELIMINARIES ON GRADED ALGEBRAS AND MODULES

We recall some basic notions and facts regarding graded algebras and modules that we will use throughout the paper.

1.1. **Graded duality.** Let k be a field and let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded k-algebra. Throughout the paper we assume Λ is *locally finite dimensional*, that is to say, we have $\dim_k \Lambda_i < \infty$ for all $i \geq 0$. It is important to note that for most of the paper we do not assume Λ_0 is semi-simple or commutative. To simplify statements of results, we will however assume throughout that Λ_0 is augmented over $k \times \cdots \times k = k^{\times r}$ for some $r > 0$, in particular we assume that Λ_0 is a basic k-algebra.

We denote by $\mathsf{Gr}\,\Lambda$ the category of graded Λ-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$ with degree 0 morphisms. We denote by $\mathsf{lfd}\,\Lambda$ the full subcategory of locally finite dimensional modules, that is, graded Λ-modules with $\dim_k M_i < \infty$ for all $i \in \mathbb{Z}$. Important full subcategories of $\mathsf{lfd}\,\Lambda$ are the category of finitely generated graded Λ-modules $\mathsf{gr}\,\Lambda$ and the category of finitely cogenerated graded Λ-modules $\mathsf{cgr}\,\Lambda$.

Forgetting the grading of Λ, we denote by $\mathsf{Mod}\,\Lambda$ the category of (ungraded) Λ-modules. If A is a finite dimensional k-algebra, then we denote by $\mathsf{mod}\,A$ the category of finitely generated (ungraded) A-modules.

If $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is a graded k-vector space, then its *graded dual* vector space DV is given by $(DV)_i = \mathrm{Hom}_k(V_{-i}, k)$. If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded Λ-module, then $DM = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_k(M_{-i}, k)$ can be given a graded Λ^{op}-module structure by $\lambda_j f_i(m) = f_i(\lambda_j m)$ for $\lambda_j \in \Lambda_j$, $f_i \in \mathrm{Hom}_k(M_{-i}, k)$ and $m \in M_{-i-j}$. If M is locally finite dimensional, then $DDM \simeq M$.

The duality D can also in a straightforward way be defined for morphisms of graded modules, and we get a duality functor $D\colon \mathsf{lfd}\,\Lambda \to \mathsf{lfd}\,\Lambda^{\mathrm{op}}$. This functor restricts to dualities $D\colon \mathsf{gr}\,\Lambda \to \mathsf{cgr}\,\Lambda^{\mathrm{op}}$ and $D\colon \mathsf{cgr}\,\Lambda \to \mathsf{gr}\,\Lambda^{\mathrm{op}}$. Restricting further to finite dimensional modules concentrated in degree zero, we get a duality $D\colon \mathsf{mod}\,\Lambda_0 \to \mathsf{mod}\,\Lambda_0^{\mathrm{op}}$. If $\Lambda = \Lambda_0$, then this is the usual duality $D\colon \mathsf{mod}\,\Lambda \to \mathsf{mod}\,\Lambda^{\mathrm{op}}$ for finite dimensional algebras.

Given a graded Λ-module M, the *jth graded shift* of M, denoted $M\langle j \rangle$, is the module with graded parts $(M\langle j \rangle)_i = M_{i-j}$ and module structure inherited from M. There is a functorial isomorphism $D(M\langle j \rangle) \simeq DM\langle -j \rangle$.

1.2. **Graded extensions.** The category $\mathsf{Gr}\,\Lambda$ is abelian, and one can define the usual homological functors. The following well-known result can for instance be found in [NV, Corollaries 2.4.4 and 2.4.7].

Lemma 1.2.1. *Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra, and let M and N be graded Λ-modules.*

(a) *If M is finitely generated, then*

$$\mathrm{Hom}_\Lambda(M, N) \simeq \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(M, N\langle j \rangle)$$

(b) *If M is finitely generated and has a projective resolution such that all syzygies are finitely generated, then*

$$\mathrm{Ext}_\Lambda^i(M, N) \simeq \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}_{\mathsf{Gr}\,\Lambda}^i(M, N\langle j \rangle)$$

for all $i \geq 0$.

With some finiteness conditions, the duality D and graded extensions combine well. A reference for the following proposition is [Mad, Proposition 1.3].

Proposition 1.2.2. *Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra, let M and N be graded locally finite dimensional Λ-modules, and let D denote the above duality. If M is bounded below or N is bounded above, then*

$$\mathrm{Ext}_{\mathsf{Gr}\,\Lambda}^i(M, N) \simeq \mathrm{Ext}_{\mathsf{Gr}\,\Lambda^{\mathrm{op}}}^i(DN, DM)$$

for all $i \geq 0$.

2. Tilting equivalence and Koszul duality

In this section we present the basic facts about tilting equivalence and Koszul duality in such a way that the similarities between the two theories are made apparent.

Let T be an object in an abelian category $\mathcal{A}$. Then $\mathsf{add}\, T$ denotes the full subcategory of $\mathcal{A}$ consisting of all finite direct sums of direct summands of T. The derived category of $\mathcal{A}$ is denoted by $\mathcal{D}(\mathcal{A})$. The *bounded derived category* of $\mathcal{A}$, that is, the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of objects with bounded homology, is denoted by $\mathcal{D}^b(\mathcal{A})$.

2.1. Tilting and cotilting modules.

Definition 2.1.1. Let A be a basic finite dimensional k-algebra and let T be a finitely generated A-module. The module T is called a *tilting module* if all of the following conditions hold.

   (i) $\operatorname{pd} T < \infty$.
   (ii) $\operatorname{Ext}_A^i(T, T) = 0$ for all $i > 0$.
   (iii) There exists a coresolution $0 \to A \to T_0 \to T_1 \to \ldots \to T_n \to 0$ with T_i in $\mathsf{add}\, T$ for all $0 \le i \le n$.

There is a dual notion of cotilting module. A finitely generated A-module U is called a *cotilting module* if (i) $\operatorname{id} U < \infty$, (ii) $\operatorname{Ext}_A^i(U, U) = 0$ for all $i > 0$ and (iii) there exists a resolution $0 \to U_m \to \ldots \to U_1 \to U_0 \to DA \to 0$ with U_i in $\mathsf{add}\, U$ for all $0 \le i \le m$.

There is a more general concept due to Wakamatsu [Wak2]. A finitely generated A-module T is called a *Wakamatsu tilting module* if (i) $\operatorname{Ext}_A^i(T, T) = 0$ for all $i > 0$ and (ii) there exists a coresolution

$$0 \to A \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \to \ldots \to T_n \to \ldots$$

with T_i in $\mathsf{add}\, T$ for all $i \ge 0$ and $\operatorname{Ext}_A^1(\ker f_i, T) = 0$ for $i > 0$. Wakamatsu cotilting modules are defined dually. It is known that a module is a Wakamatsu tilting module if and only if it is a Wakamatsu cotilting module [BS]. Tilting modules and cotilting modules are Wakamatsu tilting modules, but in general a Wakamatsu tilting module can have both infinite projective dimension and infinite injective dimension. There is the following important conjecture about Wakamatsu tilting modules with finite projective dimension.

Conjecture (Wakamatsu Tilting Conjecture [BR])**.** Let T be a Wakamatsu tilting module with finite projective dimension. Then T is a tilting module.

If A has finite global dimension, then infinite exact sequences will eventually split, and we have the following consequence.

Proposition 2.1.2. *Let A be a finite dimensional algebra with* $\operatorname{gldim} A < \infty$. *Let T be a finitely generated A-module. The following are equivalent.*

   (a) *T is a tilting module.*

(b) T *is a cotilting module.*
(c) T *is a Wakamatsu tilting module.*

For the rest of the paper we will assume that all tilting, cotilting and Wakamatsu tilting module we deal with are *basic*, that is, the indecomposable direct summands of a module are non-isomorphic. If A is augmented over $k^{\times r}$, then basic tilting or cotilting modules have r indecomposable direct summands. It is not known whether the corresponding statement holds for Wakamatsu tilting modules.

2.2. **Tilting equivalence.** Let A be a basic finite dimensional k-algebra and suppose T is a tilting A-module. Let $B = [\operatorname{End}_A(T)]^{\mathrm{op}}$. The B-module $DT \simeq \operatorname{Hom}_A(T, DA)$ is a cotilting module, and there is a ring isomorphism $A \cong [\operatorname{End}_B(DT)]^{\mathrm{op}}$. According to the Morita theory for derived categories [Ric] [KZ, chapter 6], there exists a two-sided tilting complex X and an equivalence of triangulated categories

$$\bar{G}_T = \mathbb{R}\operatorname{Hom}_A(X, -)\colon \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} B)$$

such that $H^i \bar{G}_T(M) = \operatorname{Ext}_A^i(T, M)$ for any A-module M and $i \geq 0$. A quasi-inverse of $\bar{G}_T$ is given by $\bar{F}_T = X \otimes_B^{\mathbb{L}} -\colon \mathcal{D}^b(\operatorname{mod} B) \to \mathcal{D}^b(\operatorname{mod} A)$. The functor $\bar{F}_T$ has the property that $H^{-n}\bar{F}_T(N) = \operatorname{Tor}_n^B(T, N)$ for any B-module N and $n \geq 0$.

The equivalence above restricts to an equivalence between certain module categories. Define

$$T^\perp = \{M \in \operatorname{mod} A \mid \operatorname{Ext}_A^i(T, M) = 0, \ i > 0\}$$

and

$$^\perp DT = \{N \in \operatorname{mod} B \mid \operatorname{Ext}_B^i(N, DT) = 0, \ i > 0\}.$$

We get an equivalence $\bar{E}_T = \operatorname{Hom}_A(T, -)\colon T^\perp \to {}^\perp DT$ with quasi-inverse $T \otimes_B -$.

$$
\begin{array}{ccc}
\mathcal{D}^b(\operatorname{mod} A) & \xrightarrow{\ \mathbb{R}\operatorname{Hom}_A(X,-)\ } & \mathcal{D}^b(\operatorname{mod} B) \\[2ex]
\Big\uparrow & & \Big\uparrow \\[2ex]
T^\perp & \xrightarrow{\ \operatorname{Hom}_A(T,-)\ } & {}^\perp DT
\end{array}
$$

Important objects in $T^\perp$ are T itself and the injective cogenerator DA. Important objects in $^\perp DT$ are DT and B. At the core of the equivalence $\bar{E}_T$ there is a correspondence between indecomposable summands of T and indecomposable projective B-modules and another correspondence between indecomposable injective A-modules and indecomposable summands of DT.

$$
\begin{array}{cc}
T \xmapsto{\bar{E}_T} B, & DA \xmapsto{\bar{E}_T} DT, \\[1.5ex]
T \xleftarrow[\bar{F}_T]{} B, & DA \xleftarrow[\bar{F}_T]{} DT.
\end{array}
$$

If we wish to make the correspondence more symmetrical, we can compose $\bar{E}_T D$ with the duality D and get a duality $\bar{E}_T D = \text{Hom}_{A^{\text{op}}}(-, DT)\colon {}^{\perp}_{A^{\text{op}}} DT \to {}^{\perp}_B DT$.

$$
\begin{array}{ccc}
\mathcal{D}^b(\text{mod } A) & \xrightarrow{\ \bar{G}_T\ } & \mathcal{D}^b(\text{mod } B) \\
\Big\uparrow & & \Big\uparrow \\
\end{array}
$$

$$
{}^{\perp}_{A^{\text{op}}} DT \ \xrightarrow{\ \ D\ \ }\ {}_A T^{\perp} \ \xrightarrow{\ \ \bar{E}_T\ \ }\ {}^{\perp}_B DT
$$

In this way indecomposable direct summands of DT on one side correspond to indecomposable projective modules on the other side in both directions.

$$
\begin{array}{ll}
DT \overset{\bar{E}_T D}{\longmapsto} B, & A \overset{\bar{E}_T D}{\longmapsto} DT, \\[4pt]
DT \underset{D\bar{F}_T}{\longleftarrow\!\shortmid} B, & A \underset{D\bar{F}_T}{\longleftarrow\!\shortmid} DT.
\end{array}
$$

2.3. **Koszul duality.** Koszul algebras were first defined in [Pri]. We follow here the treatment in [BGS] in which algebras are supposed to be graded with semi-simple degree zero part.

Definition 2.3.1. Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra where $\dim_k \Lambda_i < \infty$ for all $i \geq 0$ and $\Lambda_0 \cong k \times \cdots \times k = k^{\times r}$ as rings. Then Λ is called a *Koszul algebra* if $\text{Ext}^i_{\text{Gr}\,\Lambda}(\Lambda_0, \Lambda_0\langle j \rangle) = 0$ whenever $i \neq j$.

Suppose Λ is a Koszul algebra. By [BGS, Proposition 2.1.3], this is equivalent to Λ_0 having a linear projective Λ-resolution. In particular Λ_0 has a projective Λ-resolution such that all syzygies are finitely generated. Let

$$
\Gamma = [\bigoplus_{i \geq 0} \text{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)]^{\text{op}} \cong [\bigoplus_{i \geq 0} \text{Ext}^i_{\text{Gr}\,\Lambda}(\Lambda_0, \Lambda_0\langle i \rangle)]^{\text{op}}.
$$

Then Γ is also a Koszul algebra [BGS, Theorem 2.10.2], it is called the *Koszul dual* of Λ. There is an isomorphism of graded algebras

$$
\Lambda \cong [\bigoplus_{i \geq 0} \text{Ext}^i_\Gamma(\Gamma_0, \Gamma_0)]^{\text{op}} \cong [\bigoplus_{i \geq 0} \text{Ext}^i_{\text{Gr}\,\Gamma}(\Gamma_0, \Gamma_0\langle i \rangle)]^{\text{op}}.
$$

According to the theory of lifts as laid out in [Kel, 7.3 and 10.2], by lifting the set $\{\Lambda_0\langle i \rangle[i]\}_{i \in \mathbb{Z}}$, we can construct a bigraded Λ-Γ-bimodule complex X and a triangulated functor

$$
G = \mathbb{R}\,\text{Hom}_{\text{Gr}\,\Lambda}(X, -)\colon \mathcal{D}(\text{Gr}\,\Lambda) \to \mathcal{D}(\text{Gr}\,\Gamma)
$$

with the property that $(H^i G(M))_j = \text{Ext}^{i+j}_{\text{Gr}\,\Lambda}(\Lambda_0, M\langle j \rangle)$ for any graded Λ-module M and $i, j \in \mathbb{Z}$. We recall the steps of this construction in section 3.2. A left adjoint of G is given by $F = X \otimes^{\mathbb{L}}_{\text{Gr}\,\Gamma} -\colon \mathcal{D}(\text{Gr}\,\Gamma) \to \mathcal{D}(\text{Gr}\,\Lambda)$.

The functor G restricts to an equivalence between certain categories of graded modules. Define the category of *coKoszul Λ-modules*

$$
\text{cK}(\Lambda) = \{M \in \text{cgr}\,\Lambda \mid \text{Ext}^i_{\text{Gr}\,\Lambda}(\Lambda_0\langle -j \rangle, M) = 0 \text{ whenever } i \neq j\}
$$

and the category of *Koszul Γ-modules*

$$\mathsf{K}(\Gamma) = \{N \in \mathsf{gr}\,\Gamma \mid \mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(N, \Gamma_0\langle j\rangle) = 0 \text{ whenever } i \neq j\}.$$

We get an equivalence $E\colon \mathsf{cK}(\Lambda) \to \mathsf{K}(\Gamma)$, where $E = \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(\Lambda_0\langle -i\rangle, -) \simeq \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\Lambda}(\Lambda_0, -)$.

$$
\begin{array}{ccc}
\mathcal{D}(\mathsf{Gr}\,\Lambda) & \xrightarrow{\;\mathbb{R}\,\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(X,-)\;} & \mathcal{D}(\mathsf{Gr}\,\Gamma) \\
\big\uparrow & & \big\uparrow \\
\mathsf{cK}(\Lambda) & \xrightarrow{\quad E \quad} & \mathsf{K}(\Gamma)
\end{array}
$$

Important objects in $\mathsf{cK}(\Lambda)$ are Λ_0 and the graded injective cogenerator $D\Lambda$. Important objects in $\mathsf{K}(\Gamma)$ are Γ_0 and Γ. At the core of the equivalence E there is a correspondence between indecomposable graded simple Λ-modules and indecomposable projective Γ-modules and another correspondence between indecomposable graded injective Λ-modules and graded simple Γ-modules.

$$
\begin{array}{cc}
\Lambda_0 \xmapsto{\;E\;} \Gamma, & D\Lambda \xmapsto{\;E\;} \Gamma_0, \\
\Lambda_0 \xleftarrow{\;F\;} \Gamma, & D\Lambda \xleftarrow{\;F\;} \Gamma_0.
\end{array}
$$

If we wish to make the correspondence more symmetrical, we can compose E with the duality D and get a duality $ED \simeq \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(-, \Lambda_0)\colon \mathsf{K}(\Lambda^{\mathrm{op}}) \to \mathsf{K}(\Gamma)$.

$$
\begin{array}{ccc}
& \mathcal{D}(\mathsf{Gr}\,\Lambda) & \xrightarrow{\;G\;} & \mathcal{D}(\mathsf{Gr}\,\Gamma) \\
& \big\uparrow & & \big\uparrow \\
\mathsf{K}(\Lambda^{\mathrm{op}}) \xrightarrow{\;D\;} & \mathsf{cK}(\Lambda) & \xrightarrow{\;E\;} & \mathsf{K}(\Gamma)
\end{array}
$$

In this way graded simple modules on one side correspond to indecomposable projective modules on the other side in both directions.

$$
\begin{array}{cc}
\Lambda_0 \xmapsto{\;ED\;} \Gamma, & \Lambda \xmapsto{\;ED\;} \Gamma_0, \\
\Lambda_0 \xleftarrow{\;DF\;} \Gamma, & \Lambda \xleftarrow{\;DF\;} \Gamma_0.
\end{array}
$$

We often refer to the Koszul algebras of this section as *classical* Koszul algebras.

3. Graded self-orthogonal modules

We now return to graded algebras $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$, where Λ_0 is a finite dimensional algebra augmented over $k^{\times r}$, but not necessarily semi-simple. Let $1 = e_1 + \ldots + e_r$ be the decomposition of the identity in primitive orthogonal idempotents. Our aim is to describe a duality theory encompassing both

tilting equivalence and Koszul duality. In this section we look at suitable functors for such a duality theory.

3.1. **Graded and ungraded extension algebras.** Motivated by the definition of classical Koszul algebras, we define the following homological property for graded modules.

Definition 3.1.1. Let T be a finitely generated basic graded Λ-module concentrated in degree 0. We say that T is a *graded self-orthogonal module* if $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle j \rangle) = 0$ whenever $i \neq j$.

Examples of graded self-orthogonal modules are the tilting modules in the $\Lambda = \Lambda_0$ case and the module Λ_0 in the classical Koszul case.

In general such a module T might have syzygies that are not finitely generated. Even though we are not in a situation where Lemma 1.2.1 (b) applies, we can establish the following isomorphisms.

Proposition 3.1.2. *Let T be a graded self-orthogonal module. Then*
$$\operatorname{Ext}^i_\Lambda(T, T) \simeq \operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle i \rangle)$$
for each $i \geq 0$.

Proof. Suppose T has a minimal graded projective resolution
$$\mathcal{P}\colon \ \dots \to P^{-i} \to \cdots \to P^{-1} \to P^0$$

For each $i > 0$ it might happen that P^{-i} is not finitely generated, but, since T is bounded below, we always have $P^{-i} = \bigoplus_{l \in \mathbb{Z}} P^{-i}\{l\}$, where $P^{-i}\{l\}$ is a graded projective Λ-module finitely generated in degree l, for each $l \in \mathbb{Z}$. By forgetting the grading, we can regard $\mathcal{P}$ as an ungraded projective resolution of T. The complex $\operatorname{Hom}_\Lambda(\mathcal{P}, T)$ computes the groups $\operatorname{Ext}^i_\Lambda(T, T)$. The ith term of this complex is $\operatorname{Hom}_\Lambda(\bigoplus_{l \in \mathbb{Z}} P^{-i}\{l\}, T)$. For each $i \geq 0$ we have

$$
\begin{aligned}
\operatorname{Hom}_\Lambda(\bigoplus_{l \in \mathbb{Z}} P^{-i}\{l\}, T) \ &\simeq \ \prod_{l \in \mathbb{Z}} \operatorname{Hom}_\Lambda(P^{-i}\{l\}, T) \\
&\simeq \ \prod_{l \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-i}\{l\}, T\langle l \rangle),
\end{aligned}
$$

since T is concentrated in degree 0. Also, since T is concentrated in a single degree, we have

$$d(\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-i}\{l\}, T\langle l \rangle)) \subseteq \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-(i-1)}\{l\}, T\langle l \rangle)$$

for all $i > 0$, $l \in \mathbb{Z}$, where d is the differential in $\operatorname{Hom}_\Lambda(\mathcal{P}, T)$. The complex $\operatorname{Hom}_\Lambda(\mathcal{P}, T)$ is then an infinite product (indexed by l) of complexes $\operatorname{Hom}_\Lambda(\mathcal{P}\{l\}, T\langle l \rangle)$ of the form

$$\dots \to \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-i}\{l\}, T\langle l \rangle) \to \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-(i-1)}\{l\}, T\langle l \rangle) \to \dots$$
$$\dots \to \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^{-1}\{l\}, T\langle l \rangle) \to \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(P^0\{l\}, T\langle l \rangle) \to 0.$$

For each $l \in \mathbb{Z}$, since $T\langle l \rangle$ is concentrated in degree l, the complex $\mathrm{Hom}_\Lambda(\mathcal{P}\{l\}, T\langle l \rangle)$ computes the groups $\mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle l \rangle)$. We conclude that

$$\mathrm{Ext}^i_\Lambda(T, T) \simeq \prod_{l \in \mathbb{Z}} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle l \rangle)$$

for each $i \geq 0$. But $\mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle l \rangle) = 0$ whenever $i \neq l$, so

$$\mathrm{Ext}^i_\Lambda(T, T) \simeq \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle i \rangle)$$

for each $i \geq 0$. $\qquad\square$

We conclude that the graded and ungraded extension algebras are isomorphic.

Corollary 3.1.3. *Let T be a graded self-orthogonal module. There is an isomorphism of graded algebras*

$$[\bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(T, T)]^{\mathrm{op}} \cong [\bigoplus_{i \geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle i \rangle)]^{\mathrm{op}}.$$

Proof. From Proposition 3.1.2 we know there is a bijective linear map between the algebras. To prove that multiplication is preserved, we codify each algebra as a category and show that the resulting categories are equivalent. (See [Mad, section 4] for the details on how to codify a graded algebra as a category.) The graded algebra $[\bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(T, T)]^{\mathrm{op}}$ can be codified as the full subcategory $\mathcal{U} = \{T[i] \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}(\mathsf{Mod}\,\Lambda)$. The graded algebra $Ext^i_{\mathsf{Gr}\,\Lambda}(T, T\langle i \rangle)]^{\mathrm{op}}$ can be codified as the full subcategory $\mathcal{V} = \{T\langle i \rangle[i] \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}(\mathsf{Gr}\,\Lambda)$. There is a forgetful functor $|-|\colon D(\mathsf{Gr}\,\Lambda) \to D(\mathsf{Mod}\,\Lambda)$ which by (the proof of) Proposition 3.1.2 restrict to an equivalence $|-|\colon \mathcal{V} \to \mathcal{U}$. Therefore $[\bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(T, T)]^{\mathrm{op}}$ and $[\bigoplus_{i \geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle i \rangle)]^{\mathrm{op}}$ are isomorphic as graded algebras. $\qquad\square$

3.2. **Derived functors.** Let $T = \bigoplus_{h=1}^r T_{(h)}$ be a graded self-orthogonal module, where each direct summand $T_{(h)}$ is indecomposable, and let

$$\mathcal{P}\colon \ldots \to P^{-i} \to \ldots \to P^{-1} \to P^0$$

be a graded projective resolution of T. Consider the class of objects

$$\mathcal{U} = \{T_{(h)}\langle i \rangle[i] \mid 1 \leq h \leq r, i \in \mathbb{Z}\}$$

in $\mathcal{D}(\mathsf{Gr}\,\Lambda)$. Constructing the *standard lift* [Kel, 7.3] of $\mathcal{U}$, we get the graded DG algebra $\tilde{\Gamma}$ with components

$$\tilde{\Gamma}^j_i = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(P^l, P^{l+i+j}\langle i \rangle)$$

and differential

$$f \mapsto d \circ f - (-1)^n f \circ d$$

for $f \in \tilde{\Gamma}^j$, and a bigraded DG Λ-$\tilde{\Gamma}$-bimodule $\tilde{X}$ with components

$$\tilde{X}^l_{i,j} = P^{l+j}_{i-j}.$$

Let $\mathcal{D}(\tilde{\Gamma})$ denote the derived category of $\tilde{\Gamma}$. There is a pair of adjoint triangulated functors

$$\tilde{G}_T = \mathbb{R}\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\tilde{X}, -) \colon \mathcal{D}(\mathsf{Gr}\,\Lambda) \to \mathcal{D}(\tilde{\Gamma})$$

and

$$\tilde{F}_T = \tilde{X} \otimes_{\tilde{\Gamma}}^{\mathbb{L}} - \colon \mathcal{D}(\tilde{\Gamma}) \to \mathcal{D}(\mathsf{Gr}\,\Lambda).$$

By the periodicity of $\tilde{X}$, we have $\tilde{G}_T(N\langle i\rangle) = \tilde{G}_T(N)\langle -i\rangle[-i]$ for any $N \in \mathcal{D}(\mathsf{Gr}\,\Lambda)$. Since T is graded self-orthogonal, the DG algebra $\tilde{\Gamma}$ is formal and has cohomology algebra

$$\Gamma = [\bigoplus_{i \geq 0} \operatorname{Ext}_{\mathsf{Gr}\,\Lambda}^i(T, T\langle i\rangle)]^{\mathrm{op}} \cong [\bigoplus_{i \geq 0} \operatorname{Ext}_\Lambda^i(T, T)]^{\mathrm{op}}.$$

By [Kel, 9.1], we can find a bigraded DG $\tilde{\Gamma}$-Γ-bimodule Y such that

$$H_{\tilde{\Gamma}} = \mathbb{R}\operatorname{Hom}_{\tilde{\Gamma}}(Y, -) \colon \mathcal{D}(\tilde{\Gamma}) \to \mathcal{D}(\mathsf{Gr}\,\Gamma)$$

is an equivalence of triangulated categories. This equivalence commutes with the graded shift $\langle 1\rangle$. Now let $G_T = H_{\tilde{\Gamma}}\tilde{G}_T$. By [Kel, 6.3(c)], there is a complex of bigraded Λ-Γ-modules X such that

$$G_T = \mathbb{R}\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(X, -) \colon \mathcal{D}(\mathsf{Gr}\,\Lambda) \to \mathcal{D}(\mathsf{Gr}\,\Gamma).$$

This functor has a left adjoint

$$F_T = X \otimes_{\mathsf{Gr}\,\Gamma}^{\mathbb{L}} - \colon \mathcal{D}(\mathsf{Gr}\,\Gamma) \to \mathcal{D}(\mathsf{Gr}\,\Lambda).$$

We let $\phi \colon F_T G_T \to \operatorname{id}_{\mathcal{D}(\mathsf{Gr}\,\Lambda)}$ denote the counit of the adjunction.

It is perhaps fortunate that we won't be needing the actual construction of the complex X and the functor G_T. Besides being triangulated, the other important facts about G_T can be summarized in the following proposition.

Proposition 3.2.1. *Let T be a graded self-orthogonal Λ-module. Let M be a finitely cogenerated Λ-module, let N be an object in $\mathcal{D}(\mathsf{Gr}\,\Lambda)$, and let $i, j \in \mathbb{Z}$. Then*

  (a) *$G_T(T) \simeq \Gamma$.*
  (b) *The map $\phi_T \colon F_T G_T(T) \to T$ is an isomorphism.*
  (c) *There is a functorial isomorphism $G_T(N\langle j\rangle) \simeq G_T(N)\langle -j\rangle[-j]$.*
  (d) *There is a functorial isomorphism $F_T G_T(N\langle j\rangle) \simeq F_T G_T(N)\langle j\rangle$.*
  (e) *$(H^i G_T(M))_j \simeq \operatorname{Ext}_{\mathsf{Gr}\,\Lambda}^{i+j}(T, M\langle j\rangle)$.*
  (f) *$G_T(D\Lambda) \simeq DT$, where $DT \simeq \operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(T, D\Lambda)$ is considered as a graded Γ-module concentrated in degree 0.*

Proof. (a) and (b) One of the defining properties of a lift is that $\tilde{G}_T$ restricts to an equivalence between $\mathcal{U}$ and the category consisting of the indecomposable direct summands of the DG modules $\tilde{\Gamma}\langle i\rangle$, see [Kel, 7.3]. Composing with the equivalence $H_{\tilde{\Gamma}}$, we get assertions (a) and (b).

(c) By the definition of G_T it follows that

$$
\begin{aligned}
G_T(N\langle j\rangle) &= H_{\tilde{\Gamma}}\tilde{G}_T(N\langle j\rangle) \\
&= H_{\tilde{\Gamma}}(\tilde{G}_T(N)\langle -j\rangle[-j]) \\
&\simeq (H_{\tilde{\Gamma}}\tilde{G}_T(N))\langle -j\rangle[-j] \\
&= G_T(N)\langle -j\rangle[-j].
\end{aligned}
$$

(d) For any object L in $\mathcal{D}(\mathsf{Gr}\,\Lambda)$, there are isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(N\langle j\rangle), L) &\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(N\langle j\rangle), G_T(L)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(N)\langle -j\rangle[-j], G_T(L)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(N), G_T(L)\langle j\rangle[j]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(N), L\langle -j\rangle) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(N)\langle j\rangle, L).
\end{aligned}
$$

(e) There are isomorphisms

$$
\begin{aligned}
(H^i G_T(M))_j &\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(\Gamma\langle j\rangle, G_T(M)[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(T\langle -j\rangle[-j], M[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(T, M\langle j\rangle[i+j]) \\
&\simeq \mathrm{Ext}^{i+j}_{\mathsf{Gr}\,\Lambda}(T, M\langle j\rangle).
\end{aligned}
$$

(f) Since $D\Lambda$ is injective, the group $(H^i G_T(D\Lambda))_j \simeq \mathrm{Ext}^{i+j}_{\mathsf{Gr}\,\Lambda}(T, D\Lambda\langle j\rangle)$ is non-zero only when $i + j = 0$. Since T is concentrated in degree 0, we have $\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(T, D\Lambda\langle j\rangle) \neq 0$ only when $j = 0$ and as a consequence $i = 0$. Therefore, $G_T(D\Lambda) \simeq (H^0 G_T(D\Lambda))_0 \simeq \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(T, D\Lambda) \simeq DT$. $\qquad\square$

The question of whether $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism is a central one, and we will return to it in the next section.

4. T-Koszul algebras

T-Koszul algebras were first defined by Green, Reiten, and Solberg [GRS] with the purpose of developing a unified approach to Koszul duality and tilting equivalence. The original definition is rather involved, and we will not repeat the details here. In [Mad] we observed that the two central properties of T-Koszul algebras are (i) T is graded self-orthogonal as a Λ-module, and (ii) $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism. These two conditions taken together are strong enough to ensure the existence of a good duality theory. This fact led us in [Mad] to make a simplified definition and say that Λ is a T-Koszul algebra if and only if (i) and (ii) hold.

The condition on $\phi_{D\Lambda}$, while bringing much of the power, might not be so tractable, and we present here a new equivalent definition where that condition is replaced with a more elementary one. The only slightly unfortunate aspect of the new definition is that we have to assume $\mathrm{gldim}\,\Lambda_0 < \infty$, but most of the examples we are interested in are still covered.

4.1. **The new definition.**

Definition 4.1.1. Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra with gldim $\Lambda_0 < \infty$, and let T be a graded Λ-module concentrated in degree 0. We say that the algebra Λ is *T-Koszul* (or, *Koszul with respect to T*) if both of the following conditions hold.

   (i) T is a tilting Λ_0-module.
  (ii) T is graded self-orthogonal as a Λ-module.

The tilting and classical Koszul algebra cases that we want to generalize can be recovered by setting $\Lambda = \Lambda_0$ and $\Lambda_0 \cong k^{\times r}$ respectively.

If $\Lambda = \Lambda_0$, then (i) implies (ii), so Λ is Koszul with respect to T if and only if T is a tilting module.

If $\Lambda_0 \cong k^{\times r}$, then the only tilting Λ_0-module is Λ_0 itself. In this case Λ is Koszul with respect to $T = \Lambda_0$ if and only Λ is a classical Koszul algebra.

A great benefit of the new definition of T-Koszul algebras is that it is easy to establish the following important theorem. It was also obtained in [Lad] with the original definition.

Theorem 4.1.2. *Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra with* gldim $\Lambda_0 < \infty$. *Suppose Λ is a Koszul algebra with respect to a module T. Then Λ^{op} is a Koszul algebra with respect to DT.*

Proof. If T is a tilting Λ_0-module, then DT is a cotilting Λ_0^{op}-module. Since gldim $\Lambda_0^{\mathrm{op}} = $ gldim Λ_0 is finite, any cotilting Λ_0^{op}-module is also a tilting Λ_0^{op}-module.

Using the duality D we get an isomorphism

$$\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(T, T\langle j \rangle) \simeq \operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda^{\mathrm{op}}}(DT\langle -j \rangle, DT) \simeq \operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda^{\mathrm{op}}}(DT, DT\langle j \rangle).$$

So T is graded self-orthogonal as a Λ-module if and only if DT is graded self-orthogonal as a Λ^{op}-module. $\square$

If Λ is a Koszul algebra with respect to Λ_0, then Λ^{op} is a Koszul algebra with respect to $D\Lambda_0$. As the following example shows, it does not necessarily follow that Λ is a Koszul algebra with respect to $D\Lambda_0$.

Example 4.1.3. Let Λ be the path algebra given by the quiver

$$1 \xrightarrow{\ \alpha\ } 2 \xrightarrow{\ \beta\ } 3$$

and relation $\beta\alpha = 0$. Let $\deg \alpha = 2$ and $\deg \beta = 0$.

Then Λ is a Koszul algebra with respect to $\Lambda_0 = S_1 \oplus P_2 \oplus S_3$. On the other hand Λ is not a Koszul algebra with respect to $D\Lambda_0 = S_1 \oplus S_2 \oplus P_2$. Indeed, we have

$$\operatorname{Ext}^1_{\mathsf{Gr}\,\Lambda}(D\Lambda_0, D\Lambda_0\langle 2 \rangle) = \operatorname{Ext}^1_{\mathsf{Gr}\,\Lambda}(S_1, S_2\langle 2 \rangle) \neq 0,$$

and therefore $D\Lambda_0$ is not graded self-orthogonal.

The following technical proposition provides us with a class of modules M for which $G_T(M)$ is also a module and the counit map ϕ_M is an isomorphism. The module $D\Lambda$ turns out to be in this class (which we later call the class of T-*coKoszul* modules, see Theorem 4.3.1).

Proposition 4.1.4. *Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra with $\operatorname{gldim}\Lambda_0 < \infty$. Suppose Λ is a Koszul algebra with respect to a module T. Let M be a locally finite dimensional graded Λ-module which is bounded above and has the property that $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(T\langle -j \rangle, M) = 0$ whenever $i \neq j$. Then*

  (a) *$G_T(M)$ is a locally finite dimensional graded Γ-module (stalk complex concentrated in cohomological degree 0), and $G_T(M)_j = 0$ whenever $j < 0$.*
  (b) *M is a finitely cogenerated graded Λ-module, and $M_j = 0$ whenever $j > 0$.*
  (c) *The counit map $\phi_M \colon F_T G_T(M) \to M$ is an isomorphism.*
  (d) *$G_T(M)$ is a finitely generated graded Γ-module.*

Proof. (a) Since $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(T\langle -j \rangle, M) = 0$ whenever $i \neq j$, it follows from Proposition 3.2.1(e) that $(H^i G_T(M))_j \simeq \operatorname{Ext}^{i+j}_{\mathsf{Gr}\,\Lambda}(T\langle -j \rangle, M) = 0$ whenever $i \neq 0$. Since $G_T(M)$ has nonzero cohomology only in cohomological degree 0, we identify $G_T(M)$ with the graded module $H^0 G_T(M)$. We have

$$G_T(M)_j = (H^0 G_T(M))_j \simeq \operatorname{Ext}^j_{\mathsf{Gr}\,\Lambda}(T\langle -j \rangle, M)$$

for any $j \in \mathbb{Z}$, so $G_T(M)_j = 0$ whenever $j < 0$. Since T has a locally finite dimensional graded projective Λ-resolution, we have that $\operatorname{Ext}^j_{\mathsf{Gr}\,\Lambda}(T\langle -j \rangle, M)$ is finite dimensional for each $j \in \mathbb{Z}$, so $G_T(M)$ is locally finite dimensional.

(b) Since M is locally finite dimensional and bounded above, it is cogenerated by its graded socle $\operatorname{soc}_{\mathsf{gr}} M$. If $\operatorname{soc}_{\mathsf{gr}} M$ has support in degree j, then $\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\Lambda_0\langle j \rangle, M) \neq 0$. Let $0 \to \Lambda_0 \to T_0 \to T_1 \to \ldots \to T_n \to 0$ be a coresolution of Λ_0 with T_i in $\operatorname{add} T$ for all $0 \leq i \leq n$. Since $\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\Lambda_0\langle j \rangle, M) = \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(\Lambda_0\langle j \rangle, M) \neq 0$, we can find an integer $0 \leq l \leq n$ such that $\operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(T_l\langle j \rangle[-l], M) \neq 0$. Let P be the graded projective Γ-module $P = G_T(T_l)$. Since $F_T G_T(T_l) \simeq T_l$, we have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(T_l\langle j \rangle[-l], M) &\simeq \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(P\langle -j \rangle[-j - l], G_T(M)) \\
&\simeq \operatorname{Ext}^{j+l}_{\mathsf{Gr}\,\Gamma}(P\langle -j \rangle, G_T(M)).
\end{aligned}$$

Since $\operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(T_l\langle j \rangle[-l], M) \neq 0$ and P is projective, we have $j + l = 0$, and therefore $-n \leq j \leq 0$. So the degrees where $\operatorname{soc}_{\mathsf{gr}} M$ has support form a subset of $\{-n, \ldots, 0\}$. Since M is locally finite dimensional, it means that M is finitely cogenerated. We also conclude that $M_j = 0$ whenever $j > 0$.

(c) Since T is graded self-orthogonal, the map $\phi_T \colon F_T G_T(T) \to T$ is an isomorphism by Proposition 3.2.1(b). Since T is a tilting Λ_0-module, it follows from the Five Lemma that $\phi_{\Lambda_0} \colon F_T G_T(\Lambda_0) \to \Lambda_0$ is an isomorphism. Since $\operatorname{gldim}\Lambda_0 < \infty$, it also follows from the Five Lemma that $\phi_N \colon F_T G_T(N) \to N$ is an isomorphism for any finitely generated Λ_0-module

N. By Proposition 3.2.1(d), it follows that $\phi_{N\langle j\rangle}\colon F_T G_T(N\langle j\rangle) \to N\langle j\rangle$ is an isomorphism for any $j \in \mathbb{Z}$. Further use of the Five Lemma allows us to conclude that $\phi_N\colon F_T G_T(N) \to N$ is an isomorphism for any finite dimensional graded Λ-module N.

We next prove that $F_T G_T(M)$ has nonzero cohomology only in cohomological degree 0. For any $l > 0$, Proposition 3.2.1(f) implies that

$$
\begin{aligned}
D(H^i F_T G_T(M))_l &\simeq \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(M), D\Lambda\langle l\rangle[-i]) \\
&\simeq \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(M), DT\langle -l\rangle[-l-i]) \\
&= 0,
\end{aligned}
$$

since DT is concentrated in degree 0 and $G_T(M)_j = 0$ whenever $j < 0$. Now suppose $l \leq 0$ is an integer, and let $U = M_{<l}$. By Proposition 3.2.1(e), we have $(H^i G_T(U))_j \simeq \operatorname{Ext}^{i+j}_{\mathsf{Gr}\,\Lambda}(T, U\langle j\rangle)$, which is 0 whenever $j \leq -l$, since T is concentrated in degree 0. So

$$
\begin{aligned}
D(H^i F_T G_T(U))_l &\simeq \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(U), D\Lambda\langle l\rangle[-i]) \\
&\simeq \operatorname{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(U), DT\langle -l\rangle[-l-i]) \\
&= 0
\end{aligned}
$$

for all $i \in \mathbb{Z}$, since DT is concentrated in degree 0. There is a triangle

$$
F_T G_T(M_{\geq l}) \to F_T G_T(M) \to F_T G_T(U) \to F_T G_T(M_{\geq l})[1],
$$

so the cohomology long-exact sequence gives

$$
(H^i F_T G_T(M_{\geq l}))_l \simeq (H^i F_T G_T(M))_l
$$

for all $i \in \mathbb{Z}$. Since $M_{\geq l}$ is finite dimensional, the map

$$
\phi_{M_{\geq l}}\colon F_T G_T(M_{\geq l}) \to M_{\geq l}
$$

is an isomorphism. We conclude that $(H^i F_T G_T(M))_l = 0$ for all $i \neq 0$ and that $(H^0 F_T G_T(M))_l \simeq M_l$.

Since $F_T G_T(M)$ has nonzero cohomology only in cohomological degree 0, we can identify $F_T G_T(M)$ with the graded module $H^0 F_T G_T(M)$. We want to prove that $\phi_M\colon F_T G_T(M) \to M$ is an isomorphism. If $l > 0$, then $(F_T G_T(M))_l = 0 = M_l$. Suppose $l \leq 0$. We have the following morphism of triangles.

$$
\begin{array}{ccccccc}
F_T G_T(U)[-1] & \longrightarrow & F_T G_T(M_{\geq l}) & \longrightarrow & F_T G_T(M) & \longrightarrow & F_T G_T(U) \\
\Big\downarrow{\scriptstyle \phi_{U[-1]}} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle \phi_M} & & \Big\downarrow{\scriptstyle \phi_U} \\
U[-1] & \longrightarrow & M_{\geq l} & \longrightarrow & M & \longrightarrow & U.
\end{array}
$$

By taking degree l parts we get the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_l & \overset{\sim}{\longrightarrow} & (F_T G_T(M))_l & \longrightarrow & 0 \\
& & \downarrow{\wr} & & \downarrow{(\phi_M)_l} & & \\
0 & \longrightarrow & M_l & \overset{\sim}{\longrightarrow} & M_l & \longrightarrow & 0.
\end{array}
$$

So the map $(\phi_M)_l \colon (F_T G_T(M))_l \to M_l$ is an isomorphism for every $l \in \mathbb{Z}$, and therefore $\phi_M \colon F_T G_T(M) \to M$ is an isomorphism.

(d) The proof is similar to the proof of part (b). We know from part (a) that $G_T(M)$ is a locally finite dimensional graded Γ-module that is bounded below. Therefore it is generated by its graded top $\mathsf{top}_{\mathsf{gr}}(G_T(M)) = G_T(M)/J(G_T(M))$, where $J(G_T(M))$ denotes the graded Jacobson radical of $G_T(M)$. If $\mathsf{top}_{\mathsf{gr}}(G_T(M))$ has support in degree j, then

$$
\mathrm{Hom}_{\mathsf{Gr}\,\Gamma}(G_T(M), D\Gamma_0\langle j\rangle) \neq 0.
$$

By Proposition 3.2.1(f), the Γ-module $DT \simeq G_T(D\Lambda)$ is concentrated in degree 0. Considered as a module over $\Gamma_0 \cong [\mathrm{End}_\Lambda(T)]^{\mathrm{op}} \cong [\mathrm{End}_{\Lambda_0}(T)]^{\mathrm{op}}$, the module DT is a cotilting module. Let $0 \to U_m \to \ldots \to U_1 \to U_0 \to D\Gamma_0 \to 0$ be a resolution of $D\Gamma_0$ with U_i in $\mathsf{add}\, DT$ for all $0 \leq i \leq m$. Since $\mathrm{Hom}_{\mathsf{Gr}\,\Gamma}(G_T(M), D\Gamma_0\langle j\rangle) = \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(M), D\Gamma_0\langle j\rangle) \neq 0$, there is an integer $0 \leq l \leq m$ such that $\mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(M), U_l\langle j\rangle[l]) \neq 0$. Let I be the graded injective Λ-module such that $G_T(I) = U_l$. Since $F_T G_T(M) \simeq M$, we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(M), U_l\langle j\rangle[l]) & \simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(M, I\langle -j\rangle[l-j]) \\
& \simeq \mathrm{Ext}^{l-j}_{\mathsf{Gr}\,\Lambda}(M, I\langle -j\rangle).
\end{aligned}
$$

Since $\mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(M), U_l\langle j\rangle[l]) \neq 0$ and I is injective, we have $l - j = 0$, and therefore $0 \leq j \leq m$. So the degrees where $\mathsf{top}_{\mathsf{gr}}(G_T(M))$ has support form a subset of $\{0, \ldots, m\}$. Since $G_T(M)$ is locally finite dimensional, it means that M is finitely generated. $\qquad\square$

We can now prove that our new definition is equivalent to the one in [Mad], as long as $\mathrm{gldim}\,\Lambda_0 < \infty$. As a consequence the conditions in new definition are strong enough to yield a good duality theory.

Theorem 4.1.5. *Let $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ be a graded algebra with $\mathrm{gldim}\,\Lambda_0 < \infty$. Let T be a graded Λ-module concentrated in degree 0. Suppose T is graded self-orthogonal as a Λ-module. Then T is a tilting Λ_0-module if and only if the counit map $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism.*

Proof. Assume T is a tilting Λ_0-module. Since $\mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T\langle -j\rangle, D\Lambda) = 0$ except when $i = j = 0$, by Proposition 4.1.4(c) the counit map $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism.

For the converse, suppose $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism. From Proposition 3.2.1(f) we know that $G_T(D\Lambda) \simeq \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(T, D\Lambda) \simeq DT$.

Let
$$\mathcal{Q}\colon \ldots \to Q^{-(j+1)} \to Q^{-j} \to Q^{-(j-1)} \to \ldots \to Q^{-1} \to Q^0$$

be a graded projective Γ-resolution of DT. Then the complex

$$T \otimes_{\mathsf{Gr}\,\Gamma} \mathcal{Q}\colon \ldots \to T \otimes_{\mathsf{Gr}\,\Gamma} Q^{-(j+1)} \to T \otimes_{\mathsf{Gr}\,\Gamma} Q^{-j} \to T \otimes_{\mathsf{Gr}\,\Gamma} Q^{-(j-1)} \to \ldots$$
$$\ldots \to T \otimes_{\mathsf{Gr}\,\Gamma} Q^{-1} \to T \otimes_{\mathsf{Gr}\,\Gamma} Q^0$$

computes the Tor groups $\mathrm{Tor}_j^{\mathsf{Gr}\,\Gamma}(T, DT)$. Since T is concentrated in degree 0, this complex is isomorphic to

$$T \otimes_{\Gamma_0} \mathcal{Q}_0\colon \ldots \to T \otimes_{\Gamma_0}(Q^{-(j+1)})_0 \to T \otimes_{\Gamma_0}(Q^{-j})_0 \to T \otimes_{\Gamma_0}(Q^{-(j-1)})_0 \to \ldots$$
$$\ldots \to T \otimes_{\Gamma_0} (Q^{-1})_0 \to T \otimes_{\Gamma_0} (Q^0)_0,$$

where $\mathcal{Q}_0$ is the complex

$$\mathcal{Q}_0\colon \ldots \to (Q^{-(j+1)})_0 \to (Q^{-j})_0 \to (Q^{-(j-1)})_0 \to \ldots \to (Q^{-1})_0 \to (Q^0)_0.$$

This complex is a projective Λ_0-resolution of DT. We conclude that

$$\mathrm{Tor}_j^{\Gamma_0}(T, DT) \simeq \mathrm{Tor}_j^{\mathsf{Gr}\,\Gamma}(T, DT)$$

for all $j \geq 0$. Since $(\phi_{D\Lambda})_0\colon (F_T(DT))_0 \to (D\Lambda)_0$ is an isomorphism, we have $\mathrm{Tor}_j^{\mathsf{Gr}\,\Gamma}(T, DT) = 0$ for all $j \neq 0$ and $(F_T(DT))_0 \simeq T \otimes_{\mathsf{Gr}\,\Gamma} DT \simeq T \otimes_{\Gamma_0} DT \simeq (D\Lambda)_0$.

Consider the pair of adjoint functors

$$\bar{G}_T = \mathbb{R}\,\mathrm{Hom}_{\Lambda_0}(T, -)\colon \mathcal{D}^b(\mathsf{mod}\,\Lambda_0) \to \mathcal{D}^b(\mathsf{mod}\,\Gamma_0)$$

and

$$\bar{F}_T = T \otimes_{\Gamma_0}^{\mathbb{L}} -\colon \mathcal{D}^b(\mathsf{mod}\,\Gamma_0) \to \mathcal{D}^b(\mathsf{mod}\,\Lambda_0).$$

We have just shown that $\bar{\phi}_{(D\Lambda)_0}\colon \bar{F}_T\bar{G}_T((D\Lambda)_0) \simeq \bar{F}_T(DT) \to (D\Lambda)_0$ is an isomorphism, where $\bar{\phi}$ is the counit of the adjunction. Since T is a self-orthogonal Λ_0-module, it follows from [Mad, Proposition 8.3] (see also [Wak1, Proposition 5]) that T is a Wakamatsu tilting Λ_0-module. Since $\mathrm{gldim}\,\Lambda_0 < \infty$, by Proposition 2.1.2 the module T must be a tilting Λ_0-module. $\qquad\square$

4.2. **Koszul duality for algebras.** We next prove the duality theorem for T-Koszul algebras.

Theorem 4.2.1. *Let* $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ *be a graded algebra with* $\mathrm{gldim}\,\Lambda_0 < \infty$. *Suppose* Λ *is a Koszul algebra with respect to a module* T. *Then*

(a) $\mathrm{gldim}\,\Gamma_0 < \infty$, *and* Γ *is a Koszul algebra with respect to* $_\Gamma DT$.
(b) *There is an isomorphism of graded algebras*

$$\Lambda \cong [\bigoplus_{i \geq 0} \mathrm{Ext}_\Gamma^i(DT, DT)]^{\mathrm{op}} \cong [\bigoplus_{i \geq 0} \mathrm{Ext}_{\mathsf{Gr}\,\Gamma}^i(DT, DT\langle i \rangle)]^{\mathrm{op}}.$$

Proof. (a) We have $\Gamma_0 \cong [\mathrm{End}_\Lambda(T)]^{\mathrm{op}} \cong [\mathrm{End}_{\Lambda_0}(T)]^{\mathrm{op}}$. Since $\mathrm{gldim}\,\Lambda_0 < \infty$ and T is a tilting Λ_0-module, also $\mathrm{gldim}\,\Gamma_0 < \infty$. Considered as a Γ_0-module, the module $DT \simeq G_T(D\Lambda)$ is a cotilting module, and by Proposition 2.1.2 it is also a tilting module. Using Proposition 3.2.1(e), the adjointness of G_T and F_T, and Theorem 4.1.5, we get

$$
\begin{aligned}
\mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle j\rangle) &= \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(DT, DT\langle j\rangle[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_T(D\Lambda), G_T(D\Lambda)\langle j\rangle[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(F_T G_T(D\Lambda), D\Lambda\langle -j\rangle[i-j]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(D\Lambda, D\Lambda\langle -j\rangle[i-j]) \\
&\simeq \mathrm{Ext}^{i-j}_{\mathsf{Gr}\,\Lambda}(D\Lambda, D\Lambda\langle -j\rangle)
\end{aligned}
$$

Since $D\Lambda$ is injective, we have $\mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle j\rangle) = 0$ whenever $i \neq j$. So DT is a graded self-orthogonal Γ-module, and Γ is a Koszul algebra with respect to DT.

(b) For each $i \geq 0$, we have

$$
\begin{aligned}
\mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle i\rangle) &= \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(DT, DT\langle i\rangle[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(D\Lambda, D\Lambda\langle -i\rangle) \\
&\simeq \Lambda_i.
\end{aligned}
$$

These canonical isomorphisms define a bijective map

$$
\psi\colon [\bigoplus_{i\geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle i\rangle)]^{\mathrm{op}} \to \Lambda.
$$

The graded algebra $[\bigoplus_{i\geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle i\rangle)]^{\mathrm{op}}$ can be codified as the full subcategory $\mathcal{U} = \{DT\langle i\rangle[i] \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}(\mathsf{Gr}\,\Gamma)$. The graded algebra Λ can be codified as the full subcategory $\mathcal{V} = \{D\Lambda\langle -i\rangle \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}(\mathsf{Gr}\,\Lambda)$. Combining Theorem 4.1.5 with Proposition 3.2.1(d), we find that $F_T G_T(D\Lambda\langle -i\rangle) \simeq (D\Lambda\langle -i\rangle)$ for all $i \in \mathbb{Z}$. By the adjointness of F_T and G_T, the category $\mathcal{V}$ is equivalent to its essential image under G_T. By Proposition 3.2.1(c)(f) we have $G_T(D\Lambda\langle -i\rangle) \simeq DT\langle i\rangle[i]$ for all $i \in \mathbb{Z}$. So $\mathcal{U}$ and $\mathcal{V}$ are equivalent as categories. Therefore $[\bigoplus_{i\geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(DT, DT\langle i\rangle)]^{\mathrm{op}}$ and Λ are isomorphic as graded algebras. $\qquad\square$

We call the pair (Γ, DT) the *Koszul dual* of (Λ, T).

If $\Lambda = \Lambda_0$, then $\Gamma = \Gamma_0 = [\mathrm{End}_{\Lambda_0}(T)]^{\mathrm{op}}$. Similarly $[\bigoplus_{i\geq 0} \mathrm{Ext}^i_\Gamma(DT, DT)]^{\mathrm{op}} \simeq [\mathrm{End}_{\Gamma_0}(DT)]^{\mathrm{op}}$, so the theorem just states some basic facts in tilting theory.

If $\Lambda_0 \cong k^{\times r}$, the theorem above is the well known duality for classical Koszul algebras [BGS, Theorem 2.10.2].

4.3. Koszul duality for modules.

As in the case of classical Koszul algebras, the functor $G_T = \mathbb{R}\,\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(X, -)\colon \mathcal{D}(\mathsf{Gr}\,\Lambda) \to \mathcal{D}(\mathsf{Gr}\,\Gamma)$ is an equivalence only in special cases (when Λ is finite-dimensional and of finite global dimension [Mad]), but restricts in different ways to equivalences between

certain important pairs of (full) subcategories. We get the largest such equivalence if we set

$$\mathcal{T} = \{M \in \mathcal{D}(\mathsf{Gr}\,\Lambda) \mid \phi_M \colon F_T G_T(M) \to M \text{ is an isomorphism}\},$$

and let $\mathcal{S}$ be the essential image of $\mathcal{T}$ under G_T. Both $\mathcal{T}$ and $\mathcal{S}$ are closed under triangles, the translation functor, and graded shifts. By the adjointness of G_T and F_T, there is an equivalence of triangulated categories $G_T \colon \mathcal{T} \to \mathcal{S}$.

Using Proposition 4.1.4, we find the largest equivalence between categories of suitably bounded locally finite-dimensional modules given by G_T. Define the category of T-coKoszul Λ-modules

$$\mathsf{cK}_T(\Lambda) = \{M \in \mathsf{cgr}\,\Lambda \mid \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T\langle -j\rangle, M) = 0 \text{ whenever } i \neq j\}$$

and the category of T-Koszul Γ-modules

$$\mathsf{K}_T(\Gamma) = \{N \in \mathsf{gr}\,\Gamma \mid \mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(N, DT\langle j\rangle) = 0 \text{ whenever } i \neq j\}.$$

In the case $\Lambda_0 \cong k^{\times r}$, these definitions are in agreement with the classical ones.

Theorem 4.3.1. *The functor $G_T \colon \mathcal{D}(\mathsf{Gr}\,\Lambda) \to \mathcal{D}(\mathsf{Gr}\,\Gamma)$ restricts to an equivalence $E_T \colon \mathsf{cK}_T(\Lambda) \to \mathsf{K}_T(\Gamma)$, where $E_T = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda}(T\langle -i\rangle, -)$.*

$$
\begin{array}{ccc}
\mathcal{D}(\mathsf{Gr}\,\Lambda) & \xrightarrow{\;\mathbb{R}\,\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(X,-)\;} & \mathcal{D}(\mathsf{Gr}\,\Gamma) \\[2mm]
\Big\uparrow & & \Big\uparrow \\[2mm]
\mathsf{cK}_T(\Lambda) & \xrightarrow{\quad E_T \quad} & \mathsf{K}_T(\Gamma)
\end{array}
$$

Proof. If $M \in \mathsf{cK}_T(\Lambda)$, then by Proposition 4.1.4(c) the map $\phi_M \colon F_T G_T(M) \to M$ is an isomorphism. Hence $\mathsf{cK}_T(\Lambda)$ is a subcategory of $\mathcal{T}$, and therefore $\mathsf{cK}_T(\Lambda)$ is equivalent to its essential image under G_T. Let $G_T(\mathsf{cK}_T(\Lambda))$ denote this essential image. Since $F_T(G_T(M))$ is concentrated in cohomological degree 0 for any $M \in \mathsf{cK}_T(\Lambda)$, it also follows from Proposition 4.1.4 that $G_T(\mathsf{cK}_T(\Lambda))$ is a full subcategory of $\mathsf{K}_T(\Gamma)$.

Let $N \in \mathsf{K}_T(\Gamma)$. In a manner analogous to the proof of Proposition 4.1.4, we can show that $F_T(N) \in \mathsf{cK}_T(\Lambda)$ and that the unit map $\eta_N \colon N \to G_T F_T(N)$ is an isomorphism. Therefore $G_T(\mathsf{cK}_T(\Lambda)) = \mathsf{K}_T(\Gamma)$

As a consequence of Proposition 3.2.1(e), the functor G_T, when restricted to $\mathsf{cK}_T(\Lambda)$, is isomorphic to E_T. $\qquad\square$

In the case $\Lambda = \Lambda_0$, any graded Λ-module decomposes into a direct sum of its graded parts. In this case a locally finite-dimensional module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is in $\mathsf{cK}_T(\Lambda)$ if and only if (i) M is finite-dimensional, (ii) $M_i = 0$ for all $i > 0$, and (iii) $\mathrm{Ext}^j_\Lambda(T, M_i) = 0$ for all $j \neq -i$. Theorem 4.3.1 is then essentially a reformulation of [Miy, Theorem 1.16]. The category $_\Lambda T^\perp$ coincides with the full subcategory of $\mathsf{cK}_T(\Lambda)$ consisting of modules that are cogenerated in degree 0.

In earlier papers [GRS] [Mad], Koszul modules were defined to be generated in degree 0 (and by duality coKoszul modules were defined to be cogenerated in degree 0), so that one would have ${}_\Lambda T^\perp = \mathsf{cK}_T(\Lambda)$ in the $\Lambda = \Lambda_0$ case. It is important to note that this extra provision comes at a price. For general T-Koszul algebras, if all coKoszul modules M are required to be cogenerated in degree 0, then even further, more unnatural, conditions have to be put on such modules M in order to make sure that $E_T(M)$ is always generated in degree 0. The approach taken in the present paper is to make definitions as simple as possible.

In the classical Koszul case with $\Lambda_0 \cong k^{\times r}$, it follows from the definition that Koszul modules are generated in degree 0.

Important objects in $\mathsf{cK}_T(\Lambda)$ are T and the graded injective cogenerator $D\Lambda$. Important objects in $\mathsf{K}_T(\Gamma)$ are DT and Γ. At the core of the equivalence E_T there is a correspondence between indecomposable direct summands of T and indecomposable projective Γ-modules and another correspondence between indecomposable graded injective Λ-modules and indecomposable direct summands of DT.

$$T \xmapsto{E_T} \Gamma, \qquad D\Lambda \xmapsto{E_T} DT,$$
$$T \xleftarrow{\quad}_{F_T} \Gamma, \qquad D\Lambda \xleftarrow{\quad}_{F_T} DT.$$

The equivalence E_T can be composed with the duality D and we get a duality $E_T D \simeq \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\mathsf{Gr}\,\Lambda^{\mathrm{op}}}(-, DT\langle i \rangle)\colon \mathsf{K}_T(\Lambda^{\mathrm{op}}) \to \mathsf{K}_T(\Gamma)$. In this way indecomposable direct summands of DT on one side correspond to indecomposable projective modules on the other side in both directions.

$$DT \xmapsto{E_T D} \Gamma, \qquad \Lambda \xmapsto{E_T D} DT,$$
$$DT \xleftarrow{\quad}_{DF_T} \Gamma, \qquad \Lambda \xleftarrow{\quad}_{DF_T} DT.$$

The functor G_T also restricts to equivalences in the style of [MS, Theorem 2.4]. We consider here a bounded version of that theorem. Let $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ be the full subcategory of $\mathsf{gr}\,\Lambda$ consisting of modules M having a finite filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t = M$ where the factors M_i/M_{i-1} are graded shifts of direct summands of T for all $1 \leq i \leq t$. A cochain complex of graded modules $\ldots \to N^{-1} \to N^0 \to N^1 \to \ldots \to N^{j-1} \to N^j \to \ldots$ is called *linear* if N^j is generated in degree $-j$ for all $j \in \mathbb{Z}$. Let $\mathcal{L}^b(\Gamma)$ denote the category of bounded linear cochain complexes of graded projective Γ-modules and cochain maps. A homotopy map between linear complexes is always 0, so $\mathcal{L}^b(\Gamma)$ can be regarded as a full subcategory of $\mathcal{D}(\mathsf{Gr}\,\Gamma)$.

Theorem 4.3.2. *The functor $G_T\colon \mathcal{D}(\mathsf{Gr}\,\Lambda) \to \mathcal{D}(\mathsf{Gr}\,\Gamma)$ restricts to an equivalence $G_T\colon \mathcal{F}_{\mathsf{gr}\,\Lambda}(T) \to \mathcal{L}^b(\Gamma)$.*

$$
\begin{array}{ccc}
\mathcal{D}(\mathsf{Gr}\,\Lambda) & \xrightarrow{\;\mathbb{R}\,\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(X,-)\;} & \mathcal{D}(\mathsf{Gr}\,\Gamma) \\[2mm]
\Big\uparrow & & \Big\uparrow \\[2mm]
\mathcal{F}_{\mathsf{gr}\,\Lambda}(T) & \xrightarrow{\hspace{3cm}} & \mathcal{L}^b(\Gamma)
\end{array}
$$

Proof. The direct summands of T are objects in $\mathcal{T}$. Since $\mathcal{T}$ is closed under graded shifts and triangles, the category $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ is a subcategory of $\mathcal{T}$. Hence $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ is equivalent to its essential image under G_T.

If $T_{(h)}$ is a direct summand of T, then for any $i \in \mathbb{Z}$ we have that $G_T(T_{(h)}\langle i\rangle) \simeq (G_T(T_{(h)}))\langle -i\rangle[-i]$ is isomorphic to an object in $\mathcal{L}^b(\Gamma)$. Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of objects in $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$. If $G_T(L)$ and $G_T(N)$ are isomorphic to objects in $\mathcal{L}^b(\Gamma)$, then $G_T(M)$ is isomorphic to the bounded linear complex $\mathrm{cone}(G_T(N)[-1] \to G_T(L))$. By induction on the length of the filtration of M, we get that $G_T(M)$ is isomorphic to an object in $\mathcal{L}^b(\Gamma)$ for any M in $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$.

Let
$$
N\colon 0 \to N^a \to N^{a+1} \to \ldots \to N^{a+n} \to 0
$$
be a bounded linear complex of graded projective Γ-modules. The subcomplex
$$
N'\colon 0 \to N^{a+n} \to 0
$$
is isomorphic to $G_T(T'\langle a+n\rangle)$ for some T' in $\mathsf{add}\,T$. The quotient
$$
N/N'\colon 0 \to N^a \to N^{a+1} \to \ldots \to N^{a+n-1} \to 0
$$
is also a bounded linear complex. Since G_T is a triangulated functor, it follows by induction on n that $N \simeq G_T(M)$ for some M in $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$, and therefore $G_T\colon \mathcal{F}_{\mathsf{gr}\,\Lambda}(T) \to \mathcal{L}^b(\Gamma)$ is an equivalence. $\qquad\square$

We have the following corollary.

Corollary 4.3.3. *The subcategory $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ of $\mathsf{gr}\,\Lambda$ is closed under direct summands.*

Proof. Since idempotents split in $\mathcal{L}^b(\Gamma)$, the same must be true in $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$.
$\qquad\square$

If $\Lambda = \Lambda_0$, then $\mathrm{Ext}^1_\Lambda(T,T) = 0$, and the objects of $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ are finite direct sums of graded shifts of direct summands of T.

In the classical Koszul case with $\Lambda_0 \cong k^{\times r}$, the category $\mathcal{F}_{\mathsf{gr}\,\Lambda}(T)$ consists of all finite-dimensional graded Λ-modules.

If certain finiteness conditions are satisfied, we get an equivalence of bounded derived categories of finitely generated graded modules, as the following theorem shows. For classical Koszul algebras, this is [BGS, Theorem 2.12.6].

Theorem 4.3.4. *Suppose Λ is artinian and Γ is noetherian. Assume* $\operatorname{gldim}\Gamma <$
∞. *Then there is an equivalence of triangulated categories* $G_T^b \colon \mathcal{D}^b(\operatorname{gr}\Lambda) \to$
$\mathcal{D}^b(\operatorname{gr}\Gamma)$.

Proof. If Λ is artinian, then $\mathcal{D}^b(\operatorname{gr}\Lambda)$ is equivalent to the smallest triangu-
lated subcategory of $\mathcal{D}(\operatorname{Gr}\Lambda)$ containing all graded shifts of direct summands
of T. If Γ is noetherian and of finite global dimension, then $\mathcal{D}^b(\operatorname{gr}\Gamma)$ is equiv-
alent to the smallest triangulated subcategory of $\mathcal{D}(\operatorname{Gr}\Gamma)$ containing all in-
decomposable graded projective Γ-modules. Since $T \in \mathcal{T}$ and $G_T(T) = \Gamma$, it
follows from Proposition 3.2.1(c) that there is an equivalence of triangulated
categories $G_T^b \colon \mathcal{D}^b(\operatorname{gr}\Lambda) \to \mathcal{D}^b(\operatorname{gr}\Gamma)$. $\square$

4.4. **What if Λ_0 has infinite global dimension?** So far our assumption
has been that $\operatorname{gldim}\Lambda_0 < \infty$, but in this subsection we remark on the sit-
uation when $\operatorname{gldim}\Lambda_0 = \infty$. In [GRS] and [Mad], T-Koszul algebras were
defined without any condition on the global dimension of Λ_0. If, follow-
ing [Mad], we take "$\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism" to be the
defining property of T-Koszul algebras, can we find a more elementary def-
inition in the style of Definition 4.1.1 that is also appropriate in the infinite
global dimension case? As stated, the important Theorem 4.1.2 and the
possibly even more important Theorem 4.1.5 (both directions) are no longer
true if we allow $\operatorname{gldim}\Lambda_0 = \infty$. Some of the theory can be recovered by
using a crucial insight from [GRS], namely that one should allow T to be a
Wakamatsu tilting Λ_0-module.

In the special case when $\Lambda = \Lambda_0$, it is known that given a self-orthogonal
Λ-module T, the map $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism if and only
if T is a Wakamatsu tilting module (see for example [Mad, Proposition 8.3]).
So we define Λ_0 to be Koszul with respect to T if T is a Wakamatsu tilting Λ_0-
module. Can the definition for more general graded algebras be approached
in a similar way? In other words, can we determine how the appropriate
graded version of a Wakamatsu tilting module should be defined? If T
is a graded self-orthogonal Λ-module and $\phi_{D\Lambda}$ is an isomorphism, then it
follows from the proof of Theorem 4.1.5 that T is a Wakamatsu tilting Λ_0-
module, without any assumption on the global dimension of Λ_0. In line with
Definition 4.1.1, it seems natural to impose the following conditions on T.

(W1) T is a Wakamatsu tilting Λ_0-module.
(W2) T is a graded self-orthogonal Λ-module.

As natural as these conditions might be, it is our impression that (W1) and
(W2) together are not strong enough to imply that $\phi_{D\Lambda}$ is an isomorphism,
but we do not have a counterexample.

Our discussion on T-Koszul algebras with $\operatorname{gldim}\Lambda_0 = \infty$ might be related
to some important unresolved homological conjectures. We mention here
a possible connection with the Wakamatsu Tilting Conjecture. Let $\Lambda =$
$\Lambda_0 \oplus \Lambda_1$ be a graded algebra and T a Wakamatsu tilting Λ_0-module of finite
projective dimension. Then $\phi_{D\Lambda} \colon F_T G_T(D\Lambda) \to D\Lambda$ is an isomorphism if

and only if $\phi_{(D\Lambda)_{-1}}\colon F_T G_T((D\Lambda)_{-1}) \to (D\Lambda)_{-1}$ is an isomorphism. The Wakamatsu Tilting Conjecture can be reformulated as stating that for any Λ_0-module $M \in T^\perp$, the map $\phi_M\colon F_T G_T(M) \to M$ is an isomorphism. All this suggests that the following question could and should be answered in the affirmative, but we do not have a proof.

Question. Let Λ_0 be a finite dimensional k-algebra and suppose T is a Wakamatsu tilting Λ_0-module of finite projective dimension for which the Wakamatsu Tilting Conjecture does not hold. Does there exist a graded algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$ such that (i) T is a graded self-orthogonal Λ-module, (ii) $(D\Lambda)_{-1} \in T^\perp$ as a left Λ_0-module, and (iii) $\phi_{D\Lambda}\colon F_T G_T(D\Lambda) \to D\Lambda$ is not an isomorphism?

5. Quasi-hereditary Koszul algebras

The definition of T-Koszul algebras is made as unrestrictive as possible. With the almost limitless choice of algebras Λ and modules T, there is a huge variety of different kinds of algebras satisfying the requirement (see [GRS] for a wide range of examples).

Instead of taking the most general view, in this section we will have very specific examples in mind. Finite dimensional algebras arising in the representation theory of algebraic groups and Lie algebras often enjoy nice homological properties. For instance, such algebras often are Koszul and quasi-hereditary [Par]. We recall the definition of quasi-hereditary algebras below. Deeper phenomena resembling Koszul duality have been observed between algebras of this sort and extension algebras of so-called standard modules [DM]. In this section we show how the observed phenomena can be explained as instances of T-Koszul duality.

5.1. **Quasi-hereditary algebras with duality.** Let Λ be a finite dimensional k-algebra. Fix an ordering on a complete set of non-isomorphic simple Λ-modules $S_1, \ldots, S_r$. For each $0 \le i \le r$, define the *standard module* Δ_i to be the largest quotient of the projective module P_i having no simple composition factors S_j with $j > i$. Dually, define the *costandard module* ∇_i to be the largest submodule of the injective module I_i having no simple composition factors S_j with $j > i$. Let

$$\Delta = \bigoplus_{i=1}^{r} \Delta_i$$

and $\nabla = \bigoplus_{i=1}^{r} \nabla_i$.

We say that Λ is a *quasi-hereditary algebra* [CPS] [DR] if (i) Λ admits a Δ-filtration, i.e., there is a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t = \Lambda$ where the subfactors M_j/M_{j-1} are standard modules for all $1 \le j \le t$, and (ii) $\mathrm{End}_\Lambda(\Delta_i)$ is a division ring for all $1 \le i \le r$. All quasi-hereditary algebras have finite global dimension.

If Λ is quasi-hereditary, then for all $1 \leq i, j \leq r$ and $n > 0$ we have $\operatorname{Ext}_\Lambda^n(\Delta_i, \nabla_j) = 0$ [Par, Theorem 1.8]. Furthermore $\operatorname{Hom}_\Lambda(\Delta_i, \nabla_j) = 0$ whenever $i \neq j$, and $\operatorname{Hom}_\Lambda(\Delta_i, \nabla_i) \simeq \operatorname{End}(S_i)$.

A quasi-hereditary structure on an algebra determines a unique *characteristic tilting module* $\mathfrak{T}_c$, a basic tilting-cotilting module that admits both a Δ-filtration and a ∇-filtration [Rin]. For each $1 \leq i \leq r$, the module $\mathfrak{T}_c$ has an indecomposable direct summand T_i which admits morphisms $\Delta_i \hookrightarrow T_i \twoheadrightarrow \nabla_i$ with non-zero composition. These are the only indecomposable direct summands of $\mathfrak{T}_c$, so $\mathfrak{T}_c \simeq \bigoplus_{i=1}^r T_i$. The module Δ has a *tilting coresolution*

$$0 \to \Delta \to T^0 \to T^1 \to \ldots \to T^n \to 0$$

with $T^i \in \operatorname{add} \mathfrak{T}_c$ for all $0 \leq i \leq n$ [Rin]. Dually, the module ∇ has a *tilting resolution* $0 \to T^m \to \ldots \to T^1 \to T^0 \to \nabla \to 0$ with $T^i \in \operatorname{add} \mathfrak{T}_c$ for all $0 \leq i \leq m$.

In this section we only consider *quasi-hereditary algebras with duality*, that is, we suppose our quasi-hereditary algebras are equipped with a duality functor $(-)^\circ \colon \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda$ such that $S_i{}^\circ \simeq S_i$ for all $1 \leq i \leq r$. The direct summands of the characteristic tilting module have the property that $T_i{}^\circ \simeq T_i$ for all $1 \leq i \leq r$. Quasi-hereditary algebras with duality are called *BGG algebras* in [Irv]. For such algebras the BGG reciprocity principle holds [Irv]; we have

$$(P_i \colon \Delta_j) = [\Delta_j \colon S_i]$$

for all $1 \leq i, j \leq r$, where $(P_i \colon \Delta_j)$ denotes the filtration multiplicity of Δ_j in a Δ-filtration of P_i and $[\Delta_j \colon S_i]$ denotes the multiplicity of S_i in a composition series for Δ_j.

Our interest in this section lies in graded finite dimensional algebras $\Lambda = \bigoplus_{i=1}^t \Lambda_i$ that are simultaneously quasi-hereditary and (classically) Koszul. It turns out that such algebras are in many cases Koszul with respect to Δ. We begin with an example.

Example 5.1.1. Let Λ be the path algebra $\Lambda = kQ/I$, where Q is the quiver

$$1 \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^\circ}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma^\circ}{\overset{\gamma}{\rightleftarrows}} 4 \underset{\delta^\circ}{\overset{\delta}{\rightleftarrows}} 5$$

and $I = \langle \rho \rangle$ is the ideal generated by the set of relations

$$\langle \rho \rangle = \{\beta\alpha, \gamma\beta, \delta\gamma, \beta^\circ\beta - \alpha\alpha^\circ, \gamma^\circ\gamma - \beta\beta^\circ, \delta^\circ\delta - \gamma\gamma^\circ, \delta\delta^\circ, \alpha^\circ\beta^\circ, \beta^\circ\gamma^\circ, \gamma^\circ\delta^\circ\}.$$

This is a quasi-hereditary algebra with duality. It is (Morita equivalent to) a block of a Schur algebra, and it is denoted by $\Lambda_{5,0}$ in [EMM] and by $\mathcal{A}_5$ in [Erd]. The indecomposable standard modules are $\Delta_1 = S_1$,

$$\Delta_2 \colon \quad \begin{matrix} S_2 \\ {\diagup} \\ S_1 \end{matrix} \, , \quad \Delta_3 \colon \quad \begin{matrix} S_3 \\ {\diagup} \\ S_2 \end{matrix} \, , \quad \Delta_4 \colon \quad \begin{matrix} S_4 \\ {\diagup} \\ S_3 \end{matrix} \, , \quad \Delta_5 \colon \quad \begin{matrix} S_5 \\ {\diagup} \\ S_4 \end{matrix} \, .$$

The characteristic tilting module $\mathfrak{T}_c = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$ has direct summands $T_1 = S_1$,

$$
T_2\colon\ \begin{matrix} S_1 \\ \ \ \ \searrow \\ \quad\ S_2 \\ \nearrow \\ S_1 \end{matrix}
\quad,\quad
T_3\colon\ \begin{matrix} & S_2 & \\ \nearrow & & \searrow \\ S_1 & & S_3 \\ \searrow & & \nearrow \\ & S_2 & \end{matrix}
\quad,\quad
T_4\colon\ \begin{matrix} & S_3 & \\ \nearrow & & \searrow \\ S_2 & & S_4 \\ \searrow & & \nearrow \\ & S_3 & \end{matrix}
\quad,\quad
T_5\colon\ \begin{matrix} & S_4 & \\ \nearrow & & \searrow \\ S_3 & & S_5 \\ \searrow & & \nearrow \\ & S_4 & \end{matrix}
\quad.
$$

The algebra is T-Koszul in three different important ways using three different sets of orthogonal modules: namely, the indecomposable summands of the characteristic tilting module, the simple modules, and the standard modules. The grading imposed on Λ is different in each case.

If all arrows are assigned degree 0, then $\Lambda = \Lambda_0$ and Λ is Koszul with respect to $\mathfrak{T}_c$. The Koszul dual algebra of $(\Lambda, \mathfrak{T}_c)$ is the *Ringel dual* quasi-hereditary algebra $[\mathrm{End}_\Lambda(\mathfrak{T}_c)]^{\mathrm{op}}$. In this example Λ is *Ringel self-dual*, that is, there is an isomorphism $[\mathrm{End}_\Lambda(\mathfrak{T}_c)]^{\mathrm{op}} \cong \Lambda$.

If all arrows are assigned degree 1, then $\Lambda_0 \cong k^{\times r}$ and Λ is a Koszul algebra in the classical sense. The Koszul dual algebra $[\bigoplus_{i\geq 0} \mathrm{Ext}_\Lambda^i(\Lambda_0, \Lambda_0)]^{\mathrm{op}}$ is isomorphic to kQ/I', where $I' = \langle \rho' \rangle$ is the ideal generated by the set of relations

$$\langle \rho' \rangle = \{\beta^\circ \beta - \alpha \alpha^\circ, \gamma^\circ \gamma - \beta \beta^\circ, \delta^\circ \delta - \gamma \gamma^\circ, \delta \delta^\circ\}.$$

Finally, we want to show that Λ is Koszul with respect to Δ. For this we need to do a little trick with the grading. Let $\deg \alpha = \deg \beta = \deg \gamma = \deg \delta = 1$ and $\deg \alpha^\circ = \deg \beta^\circ = \deg \gamma^\circ = \deg \delta^\circ = 0$. Then Λ is Koszul with respect to $\Lambda_0 \simeq \Delta = \Delta_1 \oplus \Delta_2 \oplus \Delta_3 \oplus \Delta_4 \oplus \Delta_5$, since $\mathrm{Ext}_{\mathsf{Gr}\,\Lambda}^i(\Delta, \Delta\langle j\rangle) \neq 0$ implies $i = j$. The Koszul dual algebra $\Gamma = [\bigoplus_{i\geq 0} \mathrm{Ext}_\Lambda^i(\Delta, \Delta)]^{\mathrm{op}}$ is isomorphic to the graded algebra $k\check{Q}/\check{I}$, where $\check{Q}$ is the quiver

$$
1 \underset{\alpha^\circ}{\overset{\check{\alpha}}{\rightleftarrows}} 2 \underset{\beta^\circ}{\overset{\check{\beta}}{\rightleftarrows}} 3 \underset{\gamma^\circ}{\overset{\check{\gamma}}{\rightleftarrows}} 4 \underset{\delta^\circ}{\overset{\check{\delta}}{\rightleftarrows}} 5,
$$

the grading is given by $\deg \check{\alpha} = \deg \check{\beta} = \deg \check{\gamma} = \deg \check{\delta} = 1$ and $\deg \alpha^\circ = \deg \beta^\circ = \deg \gamma^\circ = \deg \delta^\circ = 0$, and $\check{I} = \langle \check{\rho} \rangle$ is the ideal generated by the set of relations

$$\langle \check{\rho} \rangle = \{\alpha^\circ \check{\beta} - \check{\alpha}\beta^\circ, \beta^\circ \check{\gamma} - \check{\beta}\gamma^\circ, \gamma^\circ \check{\delta} - \check{\gamma}\delta^\circ, \alpha^\circ \beta^\circ, \beta^\circ \gamma^\circ, \gamma^\circ \delta^\circ\}.$$

From Theorem 4.2.1 we know there is an isomorphism of graded algebras

$$\Lambda \cong [\bigoplus_{i\geq 0} \mathrm{Ext}_\Gamma^i(D\Delta, D\Delta)]^{\mathrm{op}}.$$

According to Theorem 4.3.4 there is an equivalence of triangulated categories

$$\mathcal{D}^b(\mathsf{gr}\,\Lambda) \to \mathcal{D}^b(\mathsf{gr}\,\Gamma).$$

It restricts to the equivalence

$$\mathcal{F}_{\mathsf{gr}\,\Lambda}(\Delta) \to \mathcal{L}^b(\Gamma)$$

from Theorem 4.3.2.

5.2. **Height functions.** The algebra in Example 5.1.1 was Koszul with respect to Δ, but not all graded quasi-hereditary algebras with duality are Koszul with respect to Δ. Sufficient conditions can be given in terms of *height functions.* A height function is a function $h\colon \{1,\ldots,r\} \to \mathbb{N}_{\geq 0}$.

Let $\Lambda = \bigoplus_{i=0}^{t} \Lambda_{(i)}$ be a graded quasi-hereditary algebra with duality. We put degree indices in parentheses to distinguish the grading from the Δ-grading below. Assume $\Lambda_{(0)} \cong k^{\times r}$. Assume further that Λ is generated in degree 1. For all $1 \leq j \leq r$, the modules Δ_j, ∇_j and T_j have graded lifts which preserve the morphisms $\Delta_j \hookrightarrow T_j \twoheadrightarrow \nabla_j$, see for example [DM]. The functor $(-)^\circ$ lifts to a duality $(-)^\circ\colon \mathsf{gr}(\Lambda) \to \mathsf{gr}(\Lambda)$ with $S_j\langle\langle l\rangle\rangle^\circ \simeq S_j\langle\langle -l\rangle\rangle$ for all $l \in \mathbb{Z}$, where $\langle\langle l\rangle\rangle$ denotes the lth graded shift for this grading.

Fix a height function h. We consider the following conditions on h.

(H1) $[(\Delta_j)_{(l)}\colon S_i] = 0$ whenever $h(i) \neq h(j) - l$.
(H2) For each $1 \leq j \leq r$, the minimal graded tilting coresolution of Δ_j,

$$0 \to \Delta_j \to T^{0,j} \to T^{1,j} \to \ldots \to T^{n,j} \to 0,$$

has the property that for all $0 \leq u \leq n$, the indecomposable direct summands of $T^{u,j}$ are of the form $T_i\langle\langle -u\rangle\rangle$ with $h(i) = h(j) - u$.

Condition (H2) is the same as condition (I) in [DM]. Via the duality $(-)^\circ$ it is also equivalent to condition (II) in that paper.

In Example 5.1.1, the height function $h(i) = i$ for all $1 \leq i \leq 5$ satisfies conditions (H1) and (H2).

We have not assumed that Λ is Koszul, but this turns out to be a consequence of (H2).

Proposition 5.2.1. *If (H2) holds, then Λ is Koszul (in the classical sense).*

Proof. From (H2) and the dual statement under $(-)^\circ$ it follows that the algebra Λ is *balanced* [Maz]. Balanced quasi-hereditary algebras are Koszul [Maz, Corollary 6]. $\qquad\square$

The existence of a height function satisfying (H1) and (H2) is more important than its actual values. The following lemma explains how values are related within the same block of the algebra.

Lemma 5.2.2. *If (H1) holds, then $e_i\Lambda_{(1)}e_j \neq 0$ implies $|h(i) - h(j)| = 1$. More precisely, suppose (H1) holds and $e_i\Lambda_{(1)}e_j \neq 0$. Then*

(a) $h(i) - h(j) = 1$ *if and only if $j < i$.*
(b) $h(j) - h(i) = 1$ *if and only if $i < j$.*

Proof. Suppose (H1) holds and $e_i\Lambda_1 e_j \neq 0$. If $i < j$, then $[(\Delta_j)_{(1)}\colon S_i] \neq 0$ and $h(i) = h(j) - 1$. If $i > j$, by using the duality $(-)^\circ$ we get that $e_j\Lambda_1 e_i \neq 0$, and by repeating the argument we get $h(j) = h(i) - 1$. We cannot have $i = j$, since $\mathrm{End}_\Lambda(\Delta_i)$ is a division ring. $\qquad\square$

In order to obtain Koszulity with respect to Δ, we follow the same strategy as in Example 5.1.1, and regrade our algebra Λ. Given a height function h satisfying (H1), we define what we call the Δ-*grading* on Λ in the following way. We assign degrees to the spaces $e_i \Lambda_{(l)} e_j$, $l \geq 0$, $1 \leq i, j \leq r$ whenever $e_i \Lambda_{(l)} e_j \neq 0$. The idempotents should be in degree 0, so we put $\deg_\Delta(e_i \Lambda_{(0)} e_i) = 0$ for all $1 \leq i \leq r$. Obviously, we have $e_i \Lambda_{(0)} e_j = 0$ whenever $i \neq j$. By assumption the algebra Λ is generated by $\Lambda_{(1)}$. With Lemma 5.2.2 in mind, we define the grading on generators by

$$\deg_\Delta(e_i \Lambda_{(1)} e_j) = \begin{cases} 1 & \text{if } h(i) - h(j) = 1 \\ 0 & \text{if } h(j) - h(i) = 1 \end{cases}$$

for all $1 \leq i, j \leq r$. This can most conveniently be expressed as

$$\deg_\Delta(e_i \Lambda_{(1)} e_j) = \frac{1 + h(i) - h(j)}{2}.$$

Hence, for all $l \geq 0$, $1 \leq i, j \leq r$ such that $e_i \Lambda_{(l)} e_j \neq 0$, we get the formula

$$\deg_\Delta(e_i \Lambda_{(l)} e_j) = \frac{l + h(i) - h(j)}{2} \geq 0.$$

The dependence of the Δ-grading on the height function is only apparent, as the next lemma shows.

Lemma 5.2.3. *As long as (H1) holds, the Δ-grading does not depend on the particular choice of height function h.*

Proof. For simplicity assume that Λ is indecomposable as an algebra. Let h and h' be two height functions satisfying (H1), and let $d = h(1) - h'(1)$. By repeated use of Lemma 5.2.2, we get that $d = h(i) - h'(i)$ for all $1 \leq i \leq r$. Then $\frac{l+h(i)-h(j)}{2} = \frac{l+h'(i)-h'(j)}{2}$ for all $l \geq 0$, $1 \leq i, j \leq r$. The general case follows. $\square$

The degree zero part of Λ under the Δ-grading is described in the following proposition.

Proposition 5.2.4. *If (H1) holds and Λ is given the Δ-grading, then $\Lambda_0 \simeq \Delta$ as graded Λ-modules.*

Proof. It is sufficient to prove that $\Lambda_0 e_j \simeq \Delta_j$ for any given $1 \leq j \leq r$. Both $\Lambda_0 e_j$ and Δ_j are quotients of $P_j = \Lambda e_j$. Let K_j denote the kernel of $P_j \twoheadrightarrow \Delta_j$. Consider an element $e_i \lambda_l e_j \in (\Lambda_{\geq 1}) e_j$ and suppose $\lambda_l \in \Lambda_{(l)}$. Then $h(i) > h(j) - l$, so $e_i \lambda_l e_j \in K_j$ by (H1). So Δ_j is a quotient of $\Lambda_0 e_j$.

Let $e_i \lambda_l e_j \in e_i \Lambda_{(l)} e_j$ with $l \geq 1$. If $l = 1$ and $e_i \lambda_1 e_j \neq 0$, then $\deg_\Delta(e_i \lambda_1 e_j) = 0$ implies $i < j$. Since the generators of Λ are in $\Lambda_{(1)}$, by induction on $l \geq 1$ it follows that $\deg_\Delta(e_i \lambda_l e_j) = 0$ implies $i < j$. The module K_j is generated by elements of the form $e_i \lambda_l e_j$ with $\lambda_l \in \Lambda_{(l)}$, $l \geq 1$ and $i > j$. Whenever $i > j$, we have $e_i \lambda_l e_j \in (\Lambda_{\geq 1}) e_j$. Therefore $K_j = (\Lambda_{\geq 1}) e_j$ and $\Lambda_0 e_j \simeq \Delta_j$. $\square$

As a consequence we have established that Δ is a tilting Λ_0-module, which is one of the conditions for Λ being Koszul with respect to Δ. The remaining

crucial step is the vanishing result in the following proposition. When (H1) holds, the minimal graded tilting coresolution of Δ in the ordinary grading,

$$0 \to \Delta \to T^0 \to T^1 \to \ldots \to T^n \to 0,$$

can be regraded according to the rule

$$\deg_\Delta(e_i T^{u,j}_{(l)}) = \frac{l + h(i) - h(j)}{2},$$

and we obtain the minimal graded tilting coresolution of $\Lambda_0 \simeq \Delta$ in the Δ-grading.

Proposition 5.2.5. *If (H1) and (H2) hold and Λ is given the Δ-grading, then* $\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, T^u\langle v \rangle) = 0$ *whenever* $u \neq v$.

Proof. Assume $\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, T^u\langle v \rangle) \neq 0$. Choose i and j such that

$$\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta_i, T^{u,j}\langle v \rangle) \neq 0.$$

By (H2), the indecomposable direct summands of $T^{u,j}$ are of the form $T_s\langle\langle -u \rangle\rangle$ with $h(s) = h(j) - u$. An ungraded morphism from Δ_i to T_s has to factor through a subfactor of T_s of the form ∇_i. According to [DM, Proposition 3.1.(ii)], the simple socle of ∇_i as a subfactor of T_s is positioned in (ordinary) degree $h(s) - h(i)$. As a subfactor of $T_s\langle\langle -u \rangle\rangle$ it is positioned in degree $l = h(s) - h(i) - u$. Since $h(s) = h(j) - u$, we get

$$\deg_\Delta(\mathrm{soc}\,\nabla_i) = \frac{l + h(i) - h(j)}{2} = -u.$$

Hence, if $\mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta_i, T_s\langle\langle -u \rangle\rangle\langle v \rangle) \neq 0$, then $u = v$, which proves the proposition. $\square$

A quasi-hereditary algebra with duality is called *multiplicity free* if $(P_i : \Delta_j) = [\Delta_j : S_i] \leq 1$ for all $1 \leq i, j \leq r$. The following example shows that (H1) and (H2) can hold even if Λ is not multiplicity free.

Example 5.2.6. Let Λ be the path algebra $\Lambda = kQ/I$, where Q is the quiver
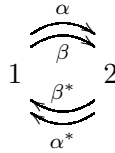
and $I = \langle \rho \rangle$ is the ideal generated by the set of relations $\langle \rho \rangle = \{\alpha\alpha^*, \beta\alpha^*, \alpha\beta^*, \beta\beta^*\}$. The indecomposable standard modules are

The characteristic tilting module $\mathfrak{T}_c = T_1 \oplus T_2$ has direct summands

$$
T_1: \quad \mathsf{S_1} \, , \qquad T_2: \quad \mathsf{S_1} \qquad\qquad \mathsf{S_1} \, .
$$

Define a height function by $h(1) = 1$ and $h(2) = 2$. Then h satisfies (H1). The standard modules have graded tilting coresolutions $0 \to \Delta_1 \to T_1 \to 0$ and

$$
0 \to \Delta_2 \to T_2 \to T_1\langle\langle -1 \rangle\rangle \oplus T_1\langle\langle -1 \rangle\rangle \to 0,
$$

so condition (H2) is also satisfied.

5.3. **Koszulity with respect to Δ.**

Theorem 5.3.1. *Let Λ be a graded quasi-hereditary algebra with duality. Let h be a height function satisfying (H1) and (H2). Regrade Λ according to the Δ-grading. Then Λ is a Koszul algebra with respect to $\Lambda_0 \simeq \Delta$.*

Proof. We have $\operatorname{gldim} \Lambda_0 \leq \operatorname{gldim} \Lambda < \infty$, and $\Lambda_0 \simeq \Delta$ is a tilting Λ_0-module. We only need to show that $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(\Delta, \Delta\langle j \rangle) = 0$ whenever $i \neq j$.

Let $0 \to \Delta \to T^0 \xrightarrow{f_0} T^1 \xrightarrow{f_1} T^2 \to \dots \to T^n \to 0$ be a minimal graded tilting coresolution of Δ. Let $K^i = \ker f_i$ for $i \geq 0$. For any $i > 0$ and $j \in \mathbb{Z}$, since $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(\Delta, T\langle j \rangle) = 0$, we get from long-exact sequences the dimension shift formula

$$
\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(\Delta, \Delta\langle j \rangle) \simeq \operatorname{Ext}^1_{\mathsf{Gr}\,\Lambda}(\Delta, K^{i-1}\langle j \rangle).
$$

There are also surjections

$$
\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, K^i\langle j \rangle) \to \operatorname{Ext}^1_{\mathsf{Gr}\,\Lambda}(\Delta, K^{i-1}\langle j \rangle) \to 0.
$$

Since K^i is a submodule of T^i and $\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, T^i\langle j \rangle) = 0$ whenever $i \neq j$ by Proposition 5.2.5, we have $\operatorname{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, K^i\langle j \rangle) = 0$ whenever $i \neq j$. So $\operatorname{Ext}^i_{\mathsf{Gr}\,\Lambda}(\Delta, \Delta\langle j \rangle) \simeq \operatorname{Ext}^1_{\mathsf{Gr}\,\Lambda}(\Delta, K^{i-1}\langle j \rangle) = 0$ whenever $i \neq j$. $\qquad\square$

If Λ and h are as in the above theorem, then in particular Δ is a graded self-orthogonal Λ-module. As usual, let $\Gamma = [\bigoplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(\Delta, \Delta)]^{\mathrm{op}}$. We construct the functor $G_\Delta \colon \mathcal{D}\,\mathsf{Gr}\,\Lambda \to \mathcal{D}\,\mathsf{Gr}\,\Gamma$ in the usual way described in section 3.2. The indecomposable graded projective Γ-modules concentrated in degree zero are of the form $G_\Delta(\Delta_i)$, $1 \leq i \leq r$.

Proposition 5.3.2. *Let Λ and h be as in Theorem 5.3.1. For any $1 \leq i \leq r$,*

$$
G_\Delta(\nabla_i) \simeq {}_\Gamma S_i,
$$

where ${}_\Gamma S_i$ is the simple top of the indecomposable graded projective Γ-module $G_\Delta(\Delta_i)$.

Proof. We have $(H^l G_\Delta(\nabla_i))_j \simeq \mathrm{Ext}^{l+j}_{\mathsf{Gr}\,\Lambda}(\Delta, \nabla_i\langle j\rangle) = 0$ whenever $j \neq 0$ or $l \neq 0$. So $G_\Delta(\nabla_i) \simeq (H^0 G_\Delta(\nabla_i))_0 \simeq \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta, \nabla_i) \simeq \mathrm{Hom}_{\mathsf{Gr}\,\Lambda}(\Delta_i, \nabla_i)$, which is 1-dimensional as a k-vector space. If $i \neq s$, then

$$\mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(G_\Delta(\Delta_s), G_\Delta(\nabla_i)) \simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(\Delta_s, \nabla_i) = 0,$$

so we must have $G_\Delta(\nabla_i) \simeq \mathrm{top}\, G_\Delta(\Delta_i)$. $\square$

When we know that Λ is Koszul with respect to Δ, we can apply all the T-Koszul machinery developed earlier in this paper and get a surprisingly clear statement of the duality theory. According to Theorem 4.2.1, the Δ-grading and the Ext-grading coincide, so the results can be stated without reference to the Δ-grading.

Corollary 5.3.3. *Let Λ be a graded quasi-hereditary algebra with duality admitting a height function h satisfying (H1) and (H2). Let*

$$\Gamma = [\bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(\Delta, \Delta)]^{\mathrm{op}}.$$

Then

$$\Lambda \cong [\bigoplus_{i \geq 0} \mathrm{Ext}^i_\Gamma(D\Delta, D\Delta)]^{\mathrm{op}}$$

as ungraded algebras. Furthermore, if Λ and Γ are given the Ext-grading, then there is an equivalence of triangulated categories $\mathcal{D}^b(\mathsf{gr}\,\Lambda) \to \mathcal{D}^b(\mathsf{gr}\,\Gamma)$ which restricts to an equivalence $\mathcal{F}_{\mathsf{gr}\,\Lambda}(\Delta) \to \mathcal{L}^b(\Gamma)$.

Proof. According to Theorem 5.3.1, the algebra Λ with the Δ-grading is Koszul with respect to Δ. Let S_u and S_v be two simple Γ-modules. Since $\dim_k \Lambda < \infty$ and $\mathrm{gldim}\,\Lambda < \infty$, the groups

$$\begin{aligned}
\mathrm{Ext}^i_{\mathsf{Gr}\,\Gamma}(S_u, S_v\langle j\rangle) &= \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Gamma}(S_u, S_v\langle j\rangle[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}\,\mathsf{Gr}\,\Lambda}(\nabla_u, \nabla_v\langle -j\rangle[i-j])
\end{aligned}$$

are non-zero only for finitely many values of i and j. So $\mathrm{gldim}\,\Gamma < \infty$. Since $\Delta \in \mathcal{D}^b(\mathsf{gr}\,\Lambda)$, the category $\mathcal{F}_{\mathsf{gr}\,\Lambda}(\Delta)$ is a subcategory of $\mathcal{D}^b(\mathsf{gr}\,\Lambda)$. The rest follows from Theorems 4.2.1, 4.3.4 and 4.3.2. $\square$

This corollary should be compared with Theorem 4.1 in [DM]. We believe our treatment of Koszul duality is more uniform and straightforward. It should be noted however that we assume the presence of a duality functor$(-)^\circ$, an assumption not made in [DM]. When Λ is a graded quasi-hereditary algebra with duality, then the conditions (I)–(IV) in [DM] imply (H1) and (H2).

It is worth writing out the statement for an important class of algebras considered in [DM], namely the algebras with module categories equivalent to blocks of the BGG category $\mathcal{O}$. Category $\mathcal{O}$ is the category of certain representations of a semi-simple Lie algebra ([Hum] is a textbook reference). The algebras arising in this way are known to be quasi-hereditary with duality. In this context the standard modules are the same as the so-called

Verma modules. The height function can be defined with the help of the Weyl group. If such an algebra is multiplicity free, the conditions (H1) and (H2) are satisfied [DM, proof of Theorem 5.1].

Corollary 5.3.4. *Let A_λ be a basic algebra corresponding to a block $\mathcal{O}_\lambda$ of category $\mathcal{O}$ for a semi-simple Lie algebra. Let Δ denote the direct sum of all Verma modules in $\mathcal{O}_\lambda$. Let $E_\lambda = [\bigoplus_{i \geq 0} \mathrm{Ext}^i_{A_\lambda}(\Delta, \Delta)]^{\mathrm{op}}$. Suppose A_λ is multiplicity free. Then $A_\lambda \cong [\bigoplus_{i \geq 0} \mathrm{Ext}^i_{E_\lambda}(D\Delta, D\Delta)]^{\mathrm{op}}$ as ungraded algebras. Furthermore, if A_λ and E_λ are given the Ext-grading, then there is an equivalence of triangulated categories $\mathcal{D}^b(\mathsf{gr}\, A_\lambda) \to \mathcal{D}^b(\mathsf{gr}\, E_\lambda)$ which restricts to an equivalence $\mathcal{F}_{\mathsf{gr}\, A_\lambda}(\Delta) \to \mathcal{L}^b(E_\lambda)$.*

REFERENCES

[BGS]   A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc., Vol. 9, No. 2, (1996) 472–527.

[BR]   A. Beligiannis, I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), no. 883, viii+207 pp.

[BS]   A. B. Buan, Ø. Solberg, *Relative cotilting theory and almost complete cotilting modules*, Algebras and modules, II (Geiranger, 1996), 77–92, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[CPS]   E. Cline, B. Parshall, L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. 391 (1988), 85–99.

[DR]   V. Dlab, C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. 33 (1989), no. 2, 280–291.

[DM]   Y. Drozd, V. Mazorchuk, *Koszul duality for extension algebras of standard modules*, J. Pure Appl. Algebra 211 (2007), no. 2, 484–496.

[Erd]   K. Erdmann, *Schur algebras of finite type.* Quart. J. Math. Oxford Ser. (2) 44 (1993), no. 173, 17–41.

[EMM]   K. Erdmann, D. Madsen, V. Miemietz, *On Auslander-Reiten translates in functorially finite subcategories and applications*, Colloq. Math. 119 (2010), 51–77.

[GRS]   E. L. Green, I. Reiten, Ø. Solberg, *Dualities on generalized Koszul algebras*, Mem. Amer. Math. Soc. 159 (2002), no. 754, xvi+67 pp.

[Hum]   J. E. Humphreys, *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, Graduate Studies in Mathematics, 94. American Mathematical Society, Providence, RI, 2008. xvi+289 pp. ISBN: 978-0-8218-4678-0

[Irv]   R. S. Irving, *BGG algebras and the BGG reciprocity principle*, J. Algebra 135 (1990), no. 2, 363–380

[Kel]   B. Keller, *Deriving DG categories*, Ann. Scient. Ec. Norm. Sup. 27 (1994), 63–102.

[KZ]   S. König, A. Zimmermann, *Derived equivalences for group rings*, Lecture Notes in Mathematics, 1685. Springer-Verlag, Berlin, 1998. x+246 pp. ISBN: 3-540-64311-7.

[Lad]   M. Lada, in preparation.

[Mad]   D. Madsen, *Derived categories and Ext algebras*, Colloq. Math. 104 (2006), 113–140.

[Maz]   V. Mazorchuk, *Koszul duality for stratified algebras. I. Balanced quasi-hereditary algebras*, Manuscripta Math. 131 (2010), no. 1-2, 1–10.

[Miy]   T. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. 193 (1986), no. 1, 113–146.

[MS]   R. Martínez-Villa, M. Saorin, *Koszul equivalences and dualities*, Pacific J. Math. 214 (2004), no. 2, 359–378.

[NV]   C. Năstăsescu, F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, 1836. Springer-Verlag, Berlin, 2004. xiv+304 pp. ISBN: 3-540-20746-5.

[Par]   B. Parshall, *Some finite-dimensional algebras arising in group theory*, Algebras and modules, I (Trondheim, 1996), 107–156, CMS Conf. Proc., 23, Amer. Math. Soc., Providence, RI, 1998.

[Pri]   S. B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 1970 39–60.

[Ric]   J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) 43 (1991), no. 1, 37–48.

[Rin]   C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (1991), no. 2, 209–223.

[Wak1] T. Wakamatsu, *On modules with trivial self-extensions*, J. Algebra 114 (1988), no. 1, 106–114.

[Wak2] T. Wakamatsu, *Stable equivalence for self-injective algebras and a generalization of tilting modules*, J. Algebra 134 (1990), no. 2, 298–325.

Department of Mathematics, 215 Carnegie, Syracuse University, Syracuse, NY 13244-1150, USA

E-mail address: dmadsen@syr.edu